Solve the following problems.

1. Suppose that $X$ and $Y$ are independent random variables with marginal density functions

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{\mu^\beta}{\Gamma(\beta)} y^{\beta-1}e^{-\mu y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$ 

In the expressions above, the Gamma function is defined by

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx.$$ 

We say that $X$ has a $\Gamma(\lambda, \alpha)$ distribution and $Y$ has a $\Gamma(\mu, \beta)$ distribution.

(a) Find the moment generating functions of $X$ and $Y$.

(b) Suppose that $\mu = \lambda$ and use these to determine the distribution of $Z = X + Y$.

2. Let $Z$ have moment generating function

$$M_Z(t) = \left(\frac{1}{2}e^{-t} + \frac{2}{5} + \frac{1}{10}e^{t/2}\right)^{36}.$$ 

Write $Z$ as a sum of i.i.d. random variables. Find the probability mass function of these random variables.

3. Suppose that $X$ and $Y$ are independent standard normal random variables. Fix $-1 < t < 1$ and let $Z = tX + \sqrt{1-t^2}Y$.

(a) Compute the correlation of $X$ and $Z$, $\rho(X, Z)$.

(b) Find the joint probability density function of $X$ and $Z$, $f_{X,Z}(x, z)$.

4. Suppose that $X$ is standard normal and $Z$ is independent of $X$ with $P(Z = 1) = P(Z = -1) = 1/2$. Let $Y = XZ$.

(a) Show that $-X$ is standard normal.

(b) Show that $Y$ is standard normal.

Hints: We cannot use our previous change of variables formulas here because $Y$ and $Z$ are not jointly continuous ($Z$ is discrete). Instead, find the CDF of $Y$ by using the law of total probability with the events $A = \{Z = 1\}$ and $A^c = \{Z = -1\}$.
(c) Show that $Cov(X, Y) = 0$.
(d) Show that if $U$ and $V$ are independent standard normal random variables, then $P(U = V) = 0$.
(e) Show that $X$ and $Y$ are not independent.

The point of this exercise is to provide a counterexample to a common misconception: even for normal random variables, being uncorrelated does not automatically imply independence.