1. Let $X$ be a uniform random variable on $[-1, 1]$. Let $Y = e^{-X}$. What is the probability density function of $Y$?

**Solution:** Since $X \in [-1, 1]$ we see that $e^{-X} \in [e^{-1}, e]$. We have

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

For $y \leq e^{-1}$,

$$P(Y \leq y) = P(e^{-X} \leq y) = 0.$$ Similarly, if $y \geq e^{1}$, then

$$P(e^{-X} \leq y) = 1.$$ If $y \in (e^{-1}, e)$, then

$$P(Y \leq y) = P(e^{-X} \leq y) = 1 - P(X \leq -\ln(y)).$$

Differentiating, we see that for $y \in (e^{-1}, e)$,

$$f_Y(y) = \frac{1}{2y}.$$

We see that

$$f_Y(y) = \begin{cases} \frac{1}{2y} & e^{-1} < y < e \\ 0 & \text{otherwise} \end{cases}.$$

2. Let $X$ be an exponential random variable with parameter $\lambda > 0$. What is the probability density function of $Y = X^2$?

**Solution:** Recall that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Then if $y < 0$, we immediately see that

$$P(Y \leq y) = P(X^2 \leq y) = 0.$$
If $y > 0$, then
\[ P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}). \]

Differentiating leads to
\[
f_Y(y) = \begin{cases} \frac{\lambda}{2\sqrt{y}} e^{-\lambda\sqrt{y}} & y > 0 \\ 0 & y \leq 0 \end{cases}
\]

3. Solve the following.

(a) (Log-normal distribution) Let $X$ be a standard normal random variable. Find the probability density function of $Y = e^X$.

(b) Let $X$ be a standard normal random variable and let $Z$ be a random variable solution of $Z^3 + Z + 1 = X$. Find the probability density function of $Z$.

Solution:

(a) Let $g(x) = e^x$. Then $g$ is strictly increasing and so has an inverse as a function from $(-\infty, \infty)$ to $(0, \infty)$, which is given by $h(x) = g^{-1}(x) = \ln(x)$. Then
\[
f_Y(y) = \begin{cases} \frac{1}{y \sqrt{2\pi}} e^{-\ln^2(y)/2} & y > 0 \\ 0 & y \leq 0 \end{cases}
\]

(b) Notice that for $h(x) = x^3 + x + 1$, $h$ is clearly continuous and we have $\lim_{x \to -\infty} h(x) = -\infty$ and $\lim_{x \to \infty} h(x) = \infty$. Moreover, since $h'(x) = 1 + x^2 > 0$, we see that $h$ is invertible. Call $g(x) = h^{-1}(x)$. Then we are looking for the probability density function of $Z = g(X)$. Since $h(x) = g^{-1}(x)$, we have for any $y \in (-\infty, \infty)$
\[
f_Y(y) = f_X(h(y)) |h'(y)| = \frac{1}{\sqrt{2\pi}} e^{-(y^3+y+1)^2/2(1+y^2)}.
\]

4. Let $X$ be a continuous random variable with probability density function given by $f_X(x) = 1/x^2$ if $x \geq 1$ and 0 otherwise. A random variable $Y$ is given by
\[
Y = \begin{cases} 2X & X \geq 2 \\ X^2 & X < 2 \end{cases}
\]

Find the probability density function of $Y$.

Solution: First we notice that the set of possible values of $Y$ is given by the set $(1, \infty)$. Define
\[
g(x) = \begin{cases} 2x & x \geq 2 \\ x^2 & 0 < x < 2 \end{cases}
\]
Then $g$ is a strictly increasing and piecewise differentiable function from $(0, \infty)$ to $(0, \infty)$ with inverse given by
\[
h(y) = \begin{cases} 
\frac{y}{2} & y \geq 4 \\
\sqrt{y} & 0 < y < 4 
\end{cases}.
\]
h is also a piecewise differentiable and strictly increasing function with
\[
h'(y) = \begin{cases} 
\frac{1}{2} & y \geq 4 \\
\frac{1}{2\sqrt{y}} & 0 < y < 4 
\end{cases}.
\]
We see then that
\[
f_Y(y) = f_X(h(y))|h'(y)| = \begin{cases} 
\frac{2}{y^2} & y \geq 4 \\
\frac{1}{2y^{3/2}} & 1 \leq y < 4 \\
0 & \text{otherwise}
\end{cases}.
\]

5. Solve the following.
(a) Let $f$ be the probability density function of a continuous random variable $X$. Find the probability density function of $Y = X^2$.
(b) Let $X$ be a standard normal random variable. Show that $Y = X^2$ has a Gamma distribution and find the parameters.

Solution:
(a) Since $X$ is continuous and $X^2 \geq 0$, we see that for $y \leq 0$, we have $P(Y \leq y) = P(X^2 \leq y) = 0$. For $y > 0$,
\[
P(Y \leq y) = P(X^2 \leq y) = P(|X| \leq \sqrt{y}) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).
\]
Differentiating, we see that
\[
f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \quad y > 0
\]
\[
f_Y(y) = 0 \quad y \leq 0.
\]
(b) If $X$ is standard normal, then
\[
f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}}y^{1/2}e^{-y/2},
\]
so $Y$ is a Gamma distribution with parameters $(1/2,1/2)$.

6. Let $X$ and $Y$ be two discrete random variables with joint mass function $f_{X,Y}(x,y)$ given by
\[
f_{X,Y}(x,y) = \begin{cases}
.4 & x = 1, y = 1 \\
.3 & x = 1, y = 2 \\
.2 & x = 2, y = 1 \\
.1 & x = 2, y = 2 \\
0 & \text{otherwise}
\end{cases}.
\]
(a) Determine if $X$ and $Y$ are independent.
(b) Compute $P(XY \leq 2)$.

**Solution:**

(a) We start by computing the marginal distributions

\[
 f_X(x) = \begin{cases} 
 .7 & x = 1 \\
 .3 & x = 2 \\
 0 & \text{otherwise}
\end{cases},
\]

\[
 f_Y(y) = \begin{cases} 
 .6 & y = 1 \\
 .4 & y = 2 \\
 0 & \text{otherwise}
\end{cases}.
\]

\[ f_{X,Y}(1,1) = .4 \neq .7 \cdot .6, \text{ so they are not independent.} \]

(b) \[ P(XY \leq 2) = 1 - P(X = 2, Y = 2) = 1 - .1 = .9. \]

7. We roll two fair dice. Let $X_1$ (resp. $X_2$) be the smallest (resp. largest) of the two outcomes.

(a) What is the joint mass function of $(X_1, X_2)$?
(b) What are the probability mass functions of $X_1$ and $X_2$?
(c) Are $X_1$ and $X_2$ independent?

**Solution:**

(a) Before computing this, we observe that the set of possible values of $(X_1, X_2)$ are given by the set \{(x_1, x_2) : x_1 \leq x_2 \text{ and } x_1, x_2 \in \{1, 2, 3, 4, 5, 6\}\}.

We do two examples before giving the result. To find $P_{X_1,X_2}(1,2)$ for example, we need to compute the probability that if we roll two fair dice, one of the dice rolls a one and one of the dice rolls a two. There are two ways to do this and each has probability 1/36. Therefore $P_{X_1,X_2}(1,2) = 1/18$. By contrast, $P(X_1, X_2)(1,1) = 1/36$ because there is only one way two roll two ones.

\[
 f_{X_1,X_2}(x_1, x_2) = \begin{cases} 
 1/18 & x_1 < x_2, \text{ and } x_1, x_2 \in \{1, 2, 3, 4, 5, 6\} \\
 1/36 & x_1 = x_2, \text{ and } x_1, x_2 \in \{1, 2, 3, 4, 5, 6\} \\
 0 & \text{otherwise}
\end{cases}.
\]

(b) To find the marginal distribution of $X_1$ and $X_2$ we compute for example

\[
 f_{X_1}(1) = f_{X_1,X_2}(1,1) + f_{X_1,X_2}(1,2) + f_{X_1,X_2}(1,3) + f_{X_1,X_2}(1,4) + f_{X_1,X_2}(1,5) + f_{X_1,X_2}(1,6) \\
 = 1/36 + 1/18 + 1/18 + 1/18 + 1/18 + 1/18 = 11/36.
\]

\[
 f_{X_1}(2) = 9/36 \\
 f_{X_1}(3) = 7/36 \\
 f_{X_1}(4) = 5/36 \\
 f_{X_1}(5) = 3/36 \\
 f_{X_1}(6) = 1/36.
\]
and \( f_{X_1}(x_1) = 0 \) for other values of \( x_1 \). Similarly,

\[
\begin{align*}
  f_{X_2}(1) &= 1/36 \\
  f_{X_2}(2) &= 3/36 \\
  f_{X_2}(3) &= 5/36 \\
  f_{X_2}(4) &= 7/36 \\
  f_{X_2}(5) &= 9/36 \\
  f_{X_2}(6) &= 11/36,
\end{align*}
\]

and \( f_{X_2}(x_2) = 0 \) for other values of \( x_2 \).

(c) No. For example \( f_{X_1}(1) \cdot f_{X_2}(1) = (11/36)(1/36) = 11/(36)^2 \neq 1/36 = f_{X_1, X_2}(1, 1) \).

8. We draw two balls with replacement out of an urn in which there are three balls numbered 2, 3, 4. Let \( X_1 \) be the sum of the outcomes and let \( X_2 \) be the product of the outcomes.

(a) What is the joint mass function of \((X_1, X_2)\)?

(b) What are the probability mass functions of \(X_1\) and \(X_2\)?

(c) Are \(X_1\) and \(X_2\) independent?

Solution:

(a) Let \(Y_1\) denote the result of the first draw and let \(Y_2\) denote the result of the second draw. Denote the joint pmf of this pair by \(f_{Y_1, Y_2}(y_1, y_2)\). Then

\[
f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 
  \frac{1}{9} & \text{if } y_1, y_2 \in \{2, 3, 4\} \\
  0 & \text{otherwise}
\end{cases}
\]

The possible values of \(X_1\) are 4, 5, 6, 7, 8 and the possible values of \(X_2\) are 4, 6, 8, 9, 12, 16. As events, we have

\[
\begin{align*}
  \{X_1 = 4\} &= \{Y_1 = 2, Y_2 = 2\} \\
  \{X_1 = 5\} &= \{Y_1 = 2, Y_2 = 3\} \cup \{Y_1 = 3, Y_2 = 2\} \\
  \{X_1 = 6\} &= \{Y_1 = 4, Y_2 = 2\} \cup \{Y_1 = 2, Y_2 = 4\} \cup \{Y_1 = 3, Y_2 = 3\} \\
  \{X_1 = 7\} &= \{Y_1 = 4, Y_2 = 3\} \cup \{Y_1 = 3, Y_2 = 4\} \\
  \{X_1 = 8\} &= \{Y_1 = 4, Y_2 = 4\} \\
  \{X_2 = 4\} &= \{Y_1 = 2, Y_2 = 2\} \\
  \{X_2 = 6\} &= \{Y_1 = 2, Y_2 = 3\} \cup \{Y_1 = 3, Y_2 = 2\} \\
  \{X_2 = 8\} &= \{Y_1 = 4, Y_2 = 2\} \cup \{Y_1 = 2, Y_2 = 4\} \\
  \{X_2 = 9\} &= \{Y_1 = 3, Y_2 = 3\} \\
  \{X_2 = 12\} &= \{Y_1 = 3, Y_2 = 4\} \cup \{Y_1 = 4, Y_2 = 3\} \\
  \{X_2 = 16\} &= \{Y_1 = 4, Y_2 = 4\}.
\end{align*}
\]

It is convenient for this problem to organize the joint pmf into a table in order to compute the marginal pmfs.
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<tr>
<th>$x_1/x_2$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>12</th>
<th>16</th>
<th>$f_{X_1}(x_1)$</th>
</tr>
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<td>0</td>
<td>3/9</td>
</tr>
</tbody>
</table>

Therefore, the pmfs are

\[
f_{X_1}(x) = \begin{cases} 
1/9 & x = 4, 8 \\
2/9 & x = 5, 7 \\
3/9 & x = 6 \\
0 & \text{otherwise} 
\end{cases},
\]

\[
f_{X_2}(x) = \begin{cases} 
1/9 & x = 4, 9, 16 \\
2/9 & x = 6, 8, 12 \\
0 & \text{otherwise} 
\end{cases}
\]

(c) No, $f_{X_1,X_2}(6, 9) = 1/9 \neq (3/9)(1/9) = f_{X_1}(6)f_{X_2}(9)$. 