MATH 5010(002) Fall 2017  
Homework 09/26/17 Solutions

Please inform your instructor if you find any errors in the solutions.

1. Is the random variable $X$ from Exercise 13.1 discrete, continuous, or neither?
   
   **Solution:** It is neither. To see that it is not discrete, notice that the collection of values it takes on is uncountably infinite (for example, $X$ can take any value in the set $[0,1)$). To see that $X$ is not continuous, notice that $P(X = 2) = 1/3 > 0$.

2. Let $X$ be a random variable with probability density function given by

   
   

   $f(x) = \begin{cases} 
   c(4 - x^2) & \text{if } -2 < x < 2 \\
   0 & \text{otherwise} 
   \end{cases}$

   (a) What is the value of $c$?

   (b) Find the cumulative distribution function of $X$.

   **Solution:**

   (a) We know that we must have $\int_{-\infty}^{\infty} f(t)dt = 1$, so

   
   $1 = \int_{-\infty}^{\infty} f(t)dt = \int_{-2}^{2} c(4 - t^2)dt = c \left(4t - \frac{t^3}{3}\right) \bigg|_{t=-2}^{t=2}$

   
   $= c \left(8 - \frac{8}{3} - (-8 - \frac{-8}{3})\right) = c(16 - \frac{16}{3})$.

   Therefore $c = 3/32$.

   (b) Notice that

   
   $\frac{3}{32} \int_{-2}^{x} 4 - t^2dt = \left. \frac{3}{32} \left(4x - \frac{x^3}{3} + 8 - \frac{8}{3}\right) \right|_{-2}^{x}$

   we see that

   

   $F(x) = \begin{cases} 
   0 & x < -2 \\
   \frac{3}{32} \left(4x - \frac{x^3}{3} + \frac{16}{3}\right) & -2 \leq x < 2 \\
   1 & x \geq 1 
   \end{cases}$

3. Let $X$ be a random variable with probability density function given by

   
   

   $f(x) = \frac{1}{2}e^{-|x|}$.

   Compute the probabilities of the following events:
(a) \{ |X| \leq 2 \}
(b) \{ |X| \leq 2 \text{ or } X \geq 0 \}
(c) \{ |X| \leq 2 \text{ or } X \leq -1 \}
(d) \{ |X| + |X - 3| \leq 3 \}
(e) \{ X^3 - X^2 - X - 2 \geq 0 \}
(f) \{ e^{\sin{(\pi X)}} \geq 1 \}
(g) \{ X \in \mathbb{N} \}

Solution:

(a) Notice that for \( t \leq 0, -|t| = t. \)

\[
P(|X| \leq 2) = P(-2 \leq X \leq 2) = \frac{1}{2} \left( \int_{-2}^{0} e^t \, dt + \int_{0}^{2} e^{-t} \, dt \right) = \int_{0}^{2} e^{-t} \, dt = 1 - e^{-2}.
\]

(b) \{ |X| \leq 2 \text{ or } X \geq 0 \} = \{ X \geq -2 \}.

\[
P(X \geq -2) = \int_{-2}^{\infty} \frac{1}{2} e^{-|t|} \, dt = \frac{1}{2} \left( \int_{-2}^{0} e^t \, dt + \int_{0}^{\infty} e^{-t} \, dt \right) = \frac{1}{2} \left( 1 - e^{-2} \right) + \frac{1}{2}.
\]

(c) \{ |X| \leq 2 \text{ or } X \leq -1 \} = \{ X \leq 2 \}.

\[
P(X \leq 2) = \int_{-\infty}^{2} \frac{1}{2} e^{-|t|} \, dt = \frac{1}{2} \left( \int_{-\infty}^{-1} e^t \, dt + \int_{-1}^{0} e^t \, dt + \int_{0}^{2} e^{-t} \, dt \right) = \frac{1}{2} + \frac{1}{2} \left( 1 - e^{-2} \right).
\]

(d) \{ |X| + |X - 3| \leq 3 \} = \{ 0 \leq X \leq 3 \}.

\[
P(0 \leq X \leq 3) = \frac{1}{2} \int_{0}^{3} e^{-t} \, dt = \frac{1}{2} (1 - e^{-3}).
\]

(e) \{ X^3 - X^2 - X - 2 \geq 0 \} = \{ (X - 2)(X^2 + X + 1) \geq 0 \} = \{ X \geq 2 \}. The first equality is factoring the polynomial \( x^3 - x^2 - x - 2 = (x - 2)(x^2 + x + 1) \). The second is observing that \( x^2 + x + 1 \geq 0 \) for all values of \( x \), so the event that \( (X - 2)(X^2 + X + 1) \geq 0 \) is the same as the event that \( X - 2 \geq 0 \).

\[
P(X \geq 2) = \int_{2}^{\infty} \frac{1}{2} e^{-t} \, dt = \frac{e^{-2}}{2}.
\]

(f) We have \{ e^{\sin{(\pi X)}} \geq 1 \} = \{ \sin{(\pi X)} \geq 0 \} = \{ X \in \cup_{k \in \mathbb{Z}} [2k, 2k + 1] \}.

\[
P(X \in \cup_{k \in \mathbb{Z}} [2k, 2k + 1]) = \sum_{k=-\infty}^{\infty} \int_{2k}^{2k+1} \frac{1}{2} e^{-|t|} \, dt
\]

\[
= \sum_{k=-\infty}^{1} \int_{2k}^{2k+1} \frac{1}{2} e^{-|t|} \, dt + \int_{0}^{1} \frac{1}{2} e^{-|t|} \, dt + \sum_{k=1}^{\infty} \int_{2k}^{2k+1} \frac{1}{2} e^{-|t|} \, dt
\]

\[
= \sum_{k=-\infty}^{1} \int_{2k}^{2k+1} \frac{1}{2} e^t \, dt + \int_{0}^{1} \frac{1}{2} e^{-t} \, dt + \sum_{k=1}^{\infty} \int_{2k}^{2k+1} \frac{1}{2} e^{-t} \, dt
\]
Now, change variables in the first sum \( j = -k \) and in the second set \( j = k \). If we also change variables in the integral \( t \mapsto -t \), we see that
\[
-1 \sum_{k=-\infty}^{\infty} \int_{2k}^{2k+1} e^t\,dt = \sum_{j=1}^{\infty} \int_{-2j}^{-2j+1} e^t\,dt = \sum_{j=1}^{\infty} \int_{2j-1}^{2j} e^{-t}\,dt
\]
But now,
\[
\sum_{j=1}^{\infty} \int_{2j-1}^{2j} e^{-t}\,dt + \int_{0}^{1} \frac{1}{2} e^{-t}\,dt + \sum_{j=1}^{\infty} \int_{2j}^{2j+1} \frac{1}{2} e^{-t}\,dt
\]
is exactly the probability that \( X \) lies in \([0, 1] \cup \bigcup_{j=1}^{\infty} [2j - 1, 2j] \cup \bigcup_{j=1}^{\infty} [2j, 2j + 1] = [0, \infty) \). Integrating as in the previous steps of the problem, we see that \( P(X \geq 0) = \frac{1}{2} \).

(g) Since \( X \) is a continuous random variable (it has a probability density function) the probability that it lies in any given discrete set (like \( \mathbb{N} \)) is zero.

4. Solve the following.

(a) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by
\[
f(x) = \begin{cases} 
\frac{c}{\sqrt{x}} & \text{if } x \geq 1, \\
0 & \text{if } x < 1
\end{cases}
\]
does there exist a value of \( c \) such that \( f \) becomes a probability density function?

(b) Let \( F : \mathbb{R} \to \mathbb{R} \) be defined by
\[
F(x) = \begin{cases} 
e^{-\frac{1}{2}} & \text{if } x > 0 \\
0 & \text{if } x \leq 0
\end{cases}
\]
Is \( F \) a cumulative distribution function? If yes, what is the associated probability density function?

Solution:

(a) Since \( f \) is piece-wise continuous and non-negative, we just need to check if there exists a value of \( c \) for which \( \int_{-\infty}^{\infty} f(t)\,dt = 1 \). But notice that
\[
\int_{1}^{n} \frac{1}{\sqrt{t}}\,dt = 2\sqrt{n} - 2,
\]
which shows that \( \int_{1}^{\infty} \frac{1}{\sqrt{t}}\,dt = \infty \), so no such \( c \) can exist.

(b) First notice that \( \lim_{x \to 0} e^{-\frac{1}{2}} = 0 \). For \( x > 0 \), \( F'(x) = \frac{1}{x}e^{-\frac{1}{2}} > 0 \), so if we recall that \( F(x) = 0 \) for \( x \leq 0 \), we can see that \( F \) is non-decreasing and that \( \lim_{x \to -\infty} F(x) = 0 \). Indeed, we can see that \( F \) is continuously differentiable, with
\[
F'(x) = \begin{cases} 
0 & x \leq 0 \\
\frac{1}{x}e^{-\frac{1}{2}} & x > 0
\end{cases}
\]
once we notice that \( \lim_{x \to 0} \frac{1}{x^2} e^{-\frac{1}{x}} = 0 \) (that this implies differentiability can be seen using the mean value theorem). In particular, we see that \( F \) is continuous and therefore right-continuous. Moreover, we see from this that \( F \) continuous and therefore right-continuous. Since \( \frac{1}{x} \to 0 \) as \( x \to \infty \), we also see that \( \lim_{x \to \infty} F(x) = 1 \), which shows that \( F \) is a distribution function. The formula for the last part of the problem is above.