1. Consider a sequence of five Bernoulli trials. Let $X$ be the number of times that a head is followed immediately by a tail. For example, if the outcome is $\omega = HHTHT$ then $X(\omega) = 2$ since a head is followed directly by a tail at trials 2 and 3 and also at trials 4 and 5. Find the probability mass function of $X$.

**Solution:** First we observe that the possible values of $X$ are 0, 1, 2. We will count out the possible configurations that give these.

In order for there to be zero heads which are followed by a tail, all of the heads that appear consecutively and at the end. The possible configurations of this form are collected together in the set $A_0 = \{\text{no heads that are followed immediately by a tails}\}$

$$A_0 = \{TTTTT, TTTTH, TTHHH, THHHH, HHHHH\}.$$  

$$P(X = 0) = (1 - p)^5 + p(1 - p)^4 + p^2(1 - p)^3 + (1 - p)^2 p^3 + (1 - p)p^4 + p^5$$

Similarly, in order for there to be one tails, we can count the ways for there to be exactly one heads which is followed by a tails, though we will not list all the elements this time.

If there are 4 heads and one tails, then we have 4 options for where to put the tails in order to have a heads before it.

Similarly if there are 4 tails and one heads, we have 4 options for where to put the heads so that there is a tails after it.

If there are two heads and three tails, then either the heads are next to each other, in which case there are three ways to place them with a tails after the second heads. If they are not next to each other, then one of the heads must occur last and (since they are not next to each other) the other cannot immediately precede it. There are three options for how this can happen. Then there are a total of 6 ways to have two heads and three tails.

If there are two tails and three heads, then either the heads are all together, in which case one tails must follow the three heads and the other tails can either be at the beginning or the end. There are then two ways for the three heads to all be together. If they are not all together, then one head must be the last flip. Either another head can be the second to last flip or the two remaining heads need to be together and they are not next to the last heads. In the former case, the tails has to go in either the second or third spot, so there are two options. If there are two heads together which are separated from the one head at the end, then either there are one or two tails between the two heads and the last tail. Adding these up, we see that there are six total ways to have two tails and three heads.

Combining all of these, we see that

$$P(X = 1) = 4p^4(1 - p) + 4p(1 - p)^4 + 6p^2(1 - p)^3 + 6p^3(1 - p)^2.$$

Finally, in order to have two heads followed immediately by tails, they must either be next to each other or separated. In any of these cases, the remaining flip can be either heads
or tails. The possible configurations are

\[ A_2 = \{HTTHT, HTHTT, HHTHT, HTHHT, HTHHT, HTTHT\}. \]

We conclude then that

\[ P(X = 2) = 3p^3(1 - p)^2 + 3p^2(1 - p)^3. \]

2. We roll a fair die three times. Let \( X \) be the number of times we roll a 6. What is the probability mass function of \( X \)?

**Solution:** If we think of rolling a 6 as a success then then \( X \) has a Binomial(3,1/6) distribution.

\[
\begin{align*}
    f(0) &= \left(\frac{5}{6}\right)^3 \\
    f(1) &= \binom{3}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2 \\
    f(2) &= \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) \\
    f(3) &= \left(\frac{1}{6}\right)^3.
\end{align*}
\]

and \( f(x) = 0 \) for all other values of \( x \).

3. We roll two fair dice.

(a) Let \( X \) be the product of the two outcomes. What is the probability mass function of \( X \)?

(b) Let \( X \) be the maximum of the two outcomes. What is the probability mass function of \( X \)?

**Solution:**

(a) By considering all products that can be formed by multiplying two numbers in \( \{1, 2, 3, 4, 5, 6\} \), we see that the possible values of \( X \) are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36. Since all pairs are equally likely, we just need to count the number of pairs that multiply together to achieve each of these.

First, we notice that in order to obtain 1, 9, 16, 25, and 36, both dice have to be the same, so there is one way to obtain each of these.

To obtain a 4, we can either roll 2 twice or one die can be equal to 1 and one can be equal to 4, so there are three ways to do this.

To obtain a 6, we can either roll a 2 and a 3 or a 6 and a 1, so there are four ways to do this. Similarly, to obtain a 12, we can either roll a 3 and a 4 or a 6 and a 2, so there are four ways to do this.

To obtain a 2, a 3, or a 5, we must roll a 1 and that number, so there are two ways to do each of these. Similarly, to obtain an 8, we must roll a 2 and a 4. To obtain a
10, we must roll a 5 and a 2. To obtain a 15, we must roll a 5 and a 3. To obtain an 18, we must roll a 6 and a 3. To roll a 20, we must roll a 5 and a 4. To obtain a 24, we must roll a 4 and a 6, and to obtain a 30, we must roll a 5 and a 6. For all of these cases, there are two ways to do this.

Combining these, we see that

\[
 f(k) = \begin{cases} 
 \frac{1}{36} & k = 1, 9, 16, 25, 36 \\
 \frac{3}{36} & k = 4 \\
 \frac{4}{36} & k = 6, 12 \\
 \frac{2}{36} & k = 2, 3, 5, 8, 10, 15, 18, 20, 24, 30 \\
 0 & \text{otherwise}
\end{cases}
\]

(b) The possible values here are 1, 2, 3, 4, 5, 6. Once again, all combinations are equally likely, so we only need to count how many combinations make each maximum possible. For \( k \in \{1, 2, 3, 4, 5, 6\} \), in order for \( k \) to be the maximum value, we need to roll \( k \) and a number less than or equal to \( k \). There is one way to roll \( k \) twice and there are \( 2(k-1) \) ways to roll \( k \) and a number strictly less than \( k \). We see that the number of combinations which result in a maximum of \( k \) are \( 1 + 2(k-1) \) and therefore for \( k \in \{1, 2, 3, 4, 5, 6\} \),

\[
 f(k) = P(X = k) = \frac{1 + 2(k-1)}{36}
\]

and for any \( x \) that is not in \( \{1, 2, 3, 4, 5, 6\} \), \( f(x) = 0 \).

4. Let \( \Omega = \{1, 2, \ldots, 6\}^2 = \{(\omega_1, \omega_2) : \omega_1, \omega_2 \in \{1, 2, \ldots, 6\}\} \) and let \( P \) be the probability measure given by \( P(\omega) = \frac{1}{36} \) for all \( \omega \in \Omega \). Let \( X : \Omega \to \mathbb{R} \) be the number of dice that rolled even. Give the probability mass function of \( X \).

**Solution:** The possible values of \( X \) are \( \{0, 1, 2\} \).

The event \( \{X = 0\} \) is the same as both rolls being odd. The event \( \{X = 1\} \) is the same as one roll being odd and one being even. The event \( \{X = 2\} \) is the same as both rolls being even. We just need to count how many ways there are to do this. The number of ways to obtain two odd numbers is the same as the number of ways to choose two numbers from the set \( \{1, 3, 5\} \) with replacement. There are \( 3 \cdot 3 = 9 \) ways to do this. The event that \( \{X = 1\} \) corresponds to picking the first dice from \( \{1, 3, 5\} \) and the second from \( \{2, 4, 6\} \) or the first from \( \{2, 4, 6\} \) and the second from \( \{1, 3, 5\} \). There are \( 3 \cdot 3 = 9 \) ways to do each of these, so there are 18 ways in total. The event \( \{X = 2\} \) corresponds to picking both from \( \{2, 4, 6\} \), so there are \( 3 \cdot 3 = 9 \) ways to do this. We find that

\[
 f(0) = \frac{9}{36} \\
 f(1) = \frac{18}{36} \\
 f(2) = \frac{9}{36}
\]

and \( f(x) = 0 \) for all other values of \( x \).
5. An urn contains 5 balls numbered from 1 to 5. We draw 3 of them at random without replacement.

(a) Let $X$ be the largest number drawn. What is the probability mass function of $X$?
(b) Let $X$ be the smallest number drawn. What is the probability mass function of $X$?

**Solution:** There are $\binom{5}{3}$ possible outcomes in this experiment. We count the number of ways for each of these to happen in order to determine the probabilities.

(a) The possible values here are 3, 4 and 5, since we are drawing without replacement. In order for $k \in \{3, 4, 5\}$ to be the minimum, we must choose $k$ and then two numbers smaller than $k$. There are $\binom{k-1}{2}$ ways to do this, so

$$f(k) = P(X = k) = \frac{\binom{k-1}{2}}{\binom{5}{3}} \text{ if } k \in \{3, 4, 6\}$$

and $f(x) = 0$ otherwise.

(b) The possible values of $k$ are $\{1, 2, 3\}$. In order for $k \in \{1, 2, 3\}$ to be the minimum, we need to pick $k$ and two numbers larger than $k$. There are $\binom{5-k}{2}$ ways to do this, so

$$f(k) = P(X = k) = \frac{\binom{5-k}{2}}{\binom{5}{3}} \text{ if } k \in \{1, 2, 3\},$$

and $f(x) = 0$ for other values of $x$.

6. Of the 100 people in a certain village, 50 always tell the truth, 30 always lie and 20 always refuse the answer. A sample size of 30 is taken with replacement.

(a) Find the probability that the sample will contain 10 people out of each category.
(b) Find the probability that there are exactly 12 liars.

**Solution:** There are $100^3 \cdot 0$ possible samples taken from this population, with replacement.

(a) This is a problem that can be solved using multinomials, as discussed in class. We will go through the argument in a bit more detail here, though.

If order mattered and we insisted that the first 10 people had to tell the truth, the second ten had to lie and the third ten had to refuse to answer, there would be $50^{10} \cdot 30^{10} \cdot 20^{10}$ ways to sample. We do not insist on this ordering, so we should multiply by the $30!$ possible permutations of these people, except that we also cannot distinguish between different people who tell the truth or those who always lie or those who refuse to answer. We have thus overcounted by a factor equal to the number of ways to permute these people, which is $10! \cdot 10! \cdot 10!$. The final answer is then that the probability is

$$\frac{30! \cdot 50^{10} \cdot 30^{10} \cdot 20^{10}}{10! \cdot 10! \cdot 10! \cdot 100^{30}}$$
(b) Similar to the previous part, if order did not matter, there would be $80^{18} \cdot 30^{12}$ ways for this to happen, but we need to account for order here. For the purposes of this part of the problem, we do not need to distinguish between those who tell the truth and those who refuse to answer, so we multiply by $\frac{30!}{18!12!}$. The probability is

$$\frac{30!}{18!12!} \cdot \frac{80^{18} \cdot 30^{12}}{100^{30}}.$$ 

7. Solve the following.

(a) Let $X$ be a geometric random variable with parameter $p$. Prove that $\sum_{n=1}^{\infty} P(X = n) = 1$.

(b) Let $Y$ be a Poisson random variable with parameter $\lambda$. Prove that $\sum_{n=0}^{\infty} P(Y = n) = 1$.

**Solution:**

(a) By the geometric series formula, for $p \in (0, 1)$,

$$\sum_{n=1}^{\infty} P(X = n) = \sum_{n=1}^{\infty} p(1-p)^{n-1} = p \sum_{j=0}^{\infty} (1-p)^j = \frac{p}{1 - (1-p)} = 1.$$ 

(b) Using the fact that for all $x \in \mathbb{R}$,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

it follows immediately that

$$\sum_{n=0}^{\infty} P(Y = n) = \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda} e^{\lambda} = 1.$$ 

8. Some day, 10000 cars are travelling across a city; one car out of 5 is gray. Suppose that the probability that a car has an accident this day is .002. Using the approximation of a binomial distribution by a Poisson distribution, compute:

(a) the probability that exactly 15 cars have an accident this day;

(b) the probability that exactly 3 gray cars have an accident this day.

**Solution:** This question should state that the proportion of cars that has an accident is the same as the proportion of gray cars that have accidents.

(a) Let $X$ denote the number of cars that have an accident. Then $X$ is $\text{Binomial}(10000, .002) = \text{Binomial}(10000, 20/10000)$. Then by the Poisson approximation,

$$P(X = 15) \approx \frac{e^{-20} 20^{15}}{15!}.$$
(b) Here we need to assume that for gray cars, the proportion that have accidents is the same as the proportion in the total population. If $Y$ denotes the number of gray cars that have an accident, $Y$ is Binomial(2000,0.002) = Binomial(2000,4/2000). Then by the Poisson approximation,

$$P(Y = 3) \approx \frac{e^{-4}4^3}{3!}.$$ 

9. Let $F$ be the function defined by

$$F(x) = \begin{cases} 
0 & x < 0 \\
\frac{x^2}{3} & 0 \leq x < 1 \\
\frac{1}{3} & 1 \leq x < 2 \\
\frac{1}{6}x + \frac{1}{3} & 2 \leq x < 4 \\
1 & x \geq 4 
\end{cases}$$

Let $X$ be a random variable which corresponds to $F$.

(a) Verify that $F$ is a cumulative distribution function.

(b) Compute $P(X = 2)$.

(c) Compute $P(X < 2)$.

(d) Compute $P(X = 2 \text{ or } \frac{1}{2} \leq x < \frac{3}{2})$.

(e) Compute $P(X = 2 \text{ or } \frac{1}{2} \leq X \leq 3)$.

Solution:

(a) There are three conditions to check.

i. To see that $F$ is non-decreasing, we first observe that it is piecewise non-decreasing on $(-\infty, 0), [0, 1), [1, 2), [2, 4)$, and $[4, \infty)$ (for example, by taking derivatives. Since the values in $(-\infty, 0)$ are all less than or equal to than $F(0) = 0$, the values in $[0, 1)$ are all less than or equal to than $F(1) = 1/3$, the values in $[1, 2)$ are less than or equal to $F(2) = 2/3$, and the values in $[2, 4)$ are all less than or equal to $F(4) = 1$, we see that $F$ is non-decreasing on all of $\mathbb{R}$.

ii. For all $x < 0, F(x) = 0$ and for all $x > 4, F(x) = 4$ so $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 4$.

iii. $F$ is clearly piecewise continuous except possibly at 0, 1, 2, 4. We can see directly that

$$\lim_{x \downarrow 0} F(x) = \lim_{x \downarrow 0} \frac{x^2}{2} = 0 = F(0).$$

$$\lim_{x \downarrow 1} F(x) = \lim_{x \downarrow 1} \frac{1}{3} = \frac{1}{3} = F(1).$$

$$\lim_{x \downarrow 2} F(x) = \lim_{x \downarrow 2} \frac{1}{6}x + \frac{1}{3} = \frac{2}{6} + \frac{1}{3} = \frac{2}{3} = F(2).$$

$$\lim_{x \downarrow 4} F(x) = \lim_{x \downarrow 4} 1 = 1 = F(4).$$
(b) 
\[ P(X = 2) = F(2) - \lim_{x \uparrow 2} F(x) = \frac{2}{3} - \lim_{x \uparrow 2} \frac{1}{3} = \frac{1}{3}. \]

(c) 
\[ P(X < 2) = \lim_{x \uparrow 2} F(x) = \lim_{x \uparrow 2} \frac{1}{3} = \frac{1}{3}. \]

(d) 
\[ P(X = 2 \text{ or } \frac{1}{2} \leq x < \frac{3}{2}) = P(X = 2) + P(\frac{1}{2} \leq x < \frac{3}{2}) \]
\[ = P(X = 2) + P(x < \frac{3}{2}) - P(X < \frac{1}{2}) \]
\[ = \frac{1}{3} + \lim_{x \uparrow \frac{3}{2}} F(x) - \lim_{x \uparrow \frac{1}{2}} F(x) \]
\[ = \frac{1}{3} + \frac{1}{3} - \frac{1}{12} = \frac{7}{12}. \]

(d) Since \( \{X = 2\} \subset \{\frac{1}{2} \leq X \leq 3\} \),
\[ P(X = 2 \text{ or } \frac{1}{2} \leq X \leq 3) = P(\frac{1}{2} \leq X \leq 3) \]
\[ = P(X \leq 3) - P(X < \frac{1}{2}) \]
\[ = F(3) - \lim_{x \uparrow \frac{1}{2}} F(x) \]
\[ = \frac{5}{6} - \frac{1}{12} \]
\[ = \frac{3}{4}. \]