Please inform your instructor if you find any errors in the solutions.

1. Suppose that we flip a fair coin. If the result of the flip is heads, we roll three dice. If
the result is tails, we roll five dice. Let \( H \) denote the number of heads we observe and \( X \)
denote the result of the number of dice which roll 2.

(a) For \( h = 1 \) and \( h = 2 \), what is the conditional pmf of \( X \) given \( H = h \), \( f_{X|H}(x|h) \)?
(b) Find the joint pmf of \( X \) and \( H \), \( f_{X,H}(x,h) \).

\[ f_{X,H}(x,h) = f_{X|H}(x|0)f_{H}(0) + f_{X|H}(x|1)f_{H}(1) \]
\[ = \begin{cases} \frac{1}{2} \left( \binom{6}{x} \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{6-x} \right) + \left( \frac{3}{2} \right) \left( \binom{3}{x} \left( \frac{1}{2} \right)^x \left( \frac{1}{2} \right)^{3-x} \right) & x = 0, 1, 2, 3 \\ 0 & x = 4, 5, 6 \end{cases} \]

2. Suppose that \( X \) and \( Y \) have joint mass function

\[ f_{X,Y}(x,y) = \begin{cases} 1/27 & (x,y) = (3,1) \\ 2/27 & (x,y) = (0,1) \\ 6/27 & (x,y) = (0,2), (1,2), (2,2), (1,3) \\ 0 & \text{otherwise.} \end{cases} \]

(a) Find the conditional mass function of \( X \) given that \( Y = 2 \), \( f_{X|Y}(x|2) \).
(b) Find the conditional expectation \( E[e^X|Y = 2] \).

\[ f_{X|Y}(x|2) = \begin{cases} \frac{1}{3} & x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases} \]
(b) \( E[e^X|Y = 2] = e^0(1/3) + e^1(1/3) + e^2(1/3) \).

3. Suppose that the joint mass function of \( X \) and \( Y \) is

\[
f_{X,Y}(x, y) = \begin{cases} 
    a_1 & x = 0, y = 0 \\
    a_2 & x = 0, y = 1 \\
    a_3 & x = 0, y = 2 \\
    \frac{1}{8} & x = 1, y = 0 \\
    a_4 & x = 1, y = 1 \\
    \frac{1}{8} & x = 1, y = 2 \\
    0 & \text{otherwise}
\end{cases}
\]

Suppose that we know the following facts:

(a) Given that \( X = 1 \), \( Y \) is uniform on \( \{0, 1, 2\} \).

(b) \( f_{X|Y}(0|0) = \frac{2}{3} \).

(c) \( E[Y|X = 0] = \frac{4}{5} \).

Using this information, find \( a_1, a_2, a_3, \) and \( a_4 \).

**Solution:** Since \( Y \) is uniform on \( \{0, 1, 2\} \) conditioned on \( X = 1 \), we must have \( f_{Y|X}(0|1) = f_{Y|X}(1|1) = f_{Y|X}(2|1) = \frac{1}{3} \). Since \( f_{Y|X}(y|1) = f_{X,Y}(1, y)/f_X(1) \), we see that we must have

\[
a_4 = \frac{1}{8}
\]

and

\[
f_X(1) = f_{X,Y}(1, 1)/f_{Y|X}(1|1) = \frac{3}{8}
\]

Since the possible values of \( X \) are 0 and 1, it follows that \( f_X(0) = \frac{5}{8} \). Then

\[
\frac{5}{8} = f_{X,Y}(0, 0) + f_{X,Y}(0, 1) + f_{X,Y}(0, 2) = a_1 + a_2 + a_3.
\]

We also have

\[
\frac{2}{3} = f_{X|Y}(0|0) = f_{X,Y}(0, 0)/f_Y(0) = (a_1)/(a_1 + 1/8)
\]

Rearranging this, we see that \( a_1 = \frac{1}{4} \). Finally, we see that

\[
f_{Y|X}1|0 = \frac{f_{X,Y}(0, 1)}{f_X(0)} = \frac{a_2}{\frac{5}{8}}
\]

and therefore

\[
4/5 = E[Y|X = 0] = 0f_{Y|X}(0|0) + 1f_{Y|X}(1|0) + 2f_{Y|X}(2|0) = 8a_2/5 + 16a_3/5.
\]
We then have the system of equations

\[ \frac{3}{8} = a_2 + a_3 \]
\[ \frac{4}{5} = 8a_2/5 + 16a_3/5 \]

which has solution \( a_2 = 1/4 \) and \( a_3 = 1/8 \). To summarize

\[ a_1 = 1/4, a_2 = 1/4, a_3 = 1/8, a_4 = 1/8. \]

4. Suppose that \( N \) is a random number which is drawn from a Binomial(10,1/4) distribution. Suppose that after observing that \( N = n \), we flip \( n \) fair coins. Let \( X \) be the number of heads that we observe from these fair coin flips.

(a) What is the conditional mass function of \( X \) given that \( N = n \), \( f_{X|N}(x|n) \)? (Be careful about the possible values of \( x \) and \( n \) here).

(b) For each possible value \( n \) of \( N \), compute \( E[X|N = n] \).

(c) Use the identity

\[ E[X] = \sum_{x,n} x f_{X,N}(x,n) = \sum_{x,n} x f_{X|N}(x|n) f_N(n) = \sum_n E[X|N = n] f_N(n) \]


to compute \( E[X] \).

Solution:

(a) The possible values of \( N \) are \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). For \( n = 1, 2, \ldots, 10, \)

\[ f_{X|N}(x|n) = \begin{cases} \binom{n}{x} (1/2)^n & x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases} \]

and for \( n = 0, \)

\[ f_{X|N}(x, 0) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases} \]

(b) Since \( X \) is conditionally Binomial(\( n, 1/2 \)) for \( n = 1, 2, \ldots, 10, \)

\[ E[X|N = n] = n/2. \]

and \( E[X|N = 0] = 0 \) (= 0/2). The last observation is just to make the formula nicer.

(c)

\[ E[X] = \sum_{n=0}^{10} E[X|N = n] f_N(n) = \sum_{n=0}^{10} (n/2) f_N(n) = E[N/2] = (1/2) E[N] = (1/2)(10)(1/4) = 5/4. \]
5. Suppose that $X_1, X_2, \text{ and } X_3$ are i.i.d. Bernoulli(1/4) random variables. Let $S_1 = X_1$, $S_2 = X_1 + X_2$ and $S_3 = X_1 + X_2 + X_3$.

(a) For all possible values of $k$, find the conditional probability mass functions of $S_3$ given $S_2 = k$ and $S_1 = k$: $f_{S_3|S_2}(x|k)$, $f_{S_3|S_1}(x|k)$.

(b) Compute $E[S_3|S_2]$ and $E[S_3|S_1]$.

**Solution:**

(a) The possible values of $S_2$ are 0, 1, 2. For $k = 0, 1, 2$,

$$f_{S_3|S_2}(x|k) = \begin{cases} 3/4 & x = k \\ 1/4 & x = k + 1 \\ 0 & \text{otherwise} \end{cases}$$

The possible values of $S_1$ are 0, 1. For $k = 0, 1$,

$$f_{S_3|S_1}(x|k) = \begin{cases} 
\binom{2}{0} (3/4)^2 & x = k \\
\binom{2}{1} (1/4)(3/4) & x = k + 1 \\
\binom{2}{2} (1/4)^2 & x = k + 2 \\
0 & \text{otherwise} \end{cases}$$

(b) From the above computations, we see that for the possible values of $k$,

$$E[S_3|S_2 = k] = k + 1/4$$
$$E[S_3|S_1 = k] = k + 1/2$$

so

$$E[S_3|S_2] = S_2 + 1/4$$
$$E[S_3|S_1] = S_1 + 1/2.$$