1. You ask a friend to choose an integer $N$ between 0 and 9. Let $A = \{ N \leq 5 \}$, $B = \{ 3 \leq N \leq 7 \}$ and $C = \{ N \text{ is even and } > 0 \}$. List the points that belong to the following events:
   a  $A \cap B \cap C$.
   b  $A \cup (B \cap C^c)$.
   c  $(A \cup B) \cap C$.
   d  $(A \cap B) \cap ((A \cup C)^c)$.

Solution: It may be helpful to list out exactly what is in each of these events:

$$A = \{ 0, 1, 2, 3, 4, 5 \}$$
$$B = \{ 3, 4, 5, 6, 7 \}$$
$$C = \{ 2, 4, 6, 8 \}.$$

a  $A \cap B = \{ 0, 1, 2, 3, 4, 5 \} \cap \{ 3, 4, 5, 6, 7 \} = \{ 3, 4, 5 \}$. Then $A \cap B \cap C = \{ 3, 4, 5 \} \cap \{ 2, 4, 6, 8 \} = \{ 4 \}$.

b  $C^c = \{ 0, 1, 3, 5, 7, 9 \}$, so $B \cap C^c = \{ 3, 4, 5, 6, 7 \} \cap \{ 0, 1, 3, 5, 7, 9 \} = \{ 3, 5, 7 \}$. Then $A \cup (B \cap C^c) = \{ 0, 1, 2, 3, 4, 5 \} \cup \{ 3, 5, 7 \} = \{ 0, 1, 2, 3, 4, 5, 7 \}$.

c  $(A \cup B) = \{ 0, 1, 2, 3, 4, 5 \} \cup \{ 3, 4, 5, 6, 7 \} = \{ 0, 1, 2, 3, 4, 5, 6, 7 \}$. $C^c = \{ 0, 1, 3, 5, 7, 9 \}$, so $(A \cup B) \cap C^c = \{ 0, 1, 3, 5, 7 \}$.

d  $A \cup C = \{ 0, 1, 2, 3, 4, 5 \} \cup \{ 2, 4, 6, 8 \} = \{ 0, 1, 2, 3, 4, 5, 6, 8 \}$, so $(A \cup C)^c = \{ 7, 9 \}$. $A \cap B = \{ 0, 1, 2, 3, 4, 5 \} \cap \{ 3, 4, 5, 6, 7 \} = \{ 3, 4, 5 \}$, which is disjoint from $\{ 7, 9 \}$. We conclude that $(A \cap B) \cap ((A \cup C)^c) = \emptyset$.

We could also do this part of the problem more easily without going through all of this. Notice that $A \cap B \subseteq A$ and $(A \cup C)^c = A^c \cap C^c$, so $(A \cup C)^c \cap A = \emptyset$.

2. Let $A$, $B$, and $C$ be arbitrary events in a sample space $\Omega$. Express each of the following events in terms of $A$, $B$, and $C$ using intersections, unions, and complements.

   a  $A$ and $B$ occur, but not $C$;
   b  $A$ is the only one to occur;
   c  at least two of the events $A$, $B$, $C$ occur;
   d  at least one of the events $A$, $B$, $C$ occurs;
   e  exactly two of the events $A$, $B$, $C$ occur;
   f  exactly one of the events $A$, $B$, $C$ occurs;
   g  not more than one of the events $A$, $B$, $C$ occur.
Solution:

a \( A \cap B \cap C^c. \)
b \( A \cap B^c \cap C^c. \)
c \( (A \cap B) \cup (B \cap C) \cup (A \cap C). \) This is also the same as \( (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C^c). \)
d \( A \cup B \cup C. \) As above, you could also enumerate all the possibilities here, but it would be messier.
f \( (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B \cap C^c). \)
g \( (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A \cap B^c \cap C^c). \)

3. Two sets are disjoint if their intersection is empty. If \( A \) and \( B \) are disjoint events in a sample space \( \Omega \), are \( A^c \) and \( B^c \) disjoint? Are \( A \cap C \) and \( B \cap C \) disjoint? What about \( A \cup C \) and \( B \cup C \)?

Solution:

a Take the example of \( \Omega = \{1, 2, 3\} \) and let \( A = \{1\} \) and \( B = \{3\} \). Then \( A^c = \{2, 3\} \) and \( B^c = \{1, 2\} \), so \( A^c \) and \( B^c \) are not disjoint.
b If \( A \) and \( B \) are disjoint, then \( (A \cap C) \cap (B \cap C) = A \cap B \cap C = \emptyset \cap C = \emptyset. \)
c In the same setting as above, let \( C = \{2\} \), so \( (A \cup C) \cap (B \cup C) = \{2\} = C. \)

4. An urn contains three chips: one black, one green, and one red. We draw one chip at random. Give a sample space \( \Omega \) and a collection of events \( F \) for this experiment.

Solution: We will denote the event that the chip is black by \( B \), the event that it is green by \( G \), and the event that it is red by \( R \). We can then choose

\[
\Omega = \{B, G, R\}
\]

\[
F = \{\emptyset, B, G, R, \{B, G\}, \{B, R\}, \{G, R\}, \{B, G, R\}\}.
\]

This choice of \( F \) is the usual one in the setting where we only have finitely many possible outcomes: it is the collection of all subsets of \( \Omega \).

5. A public opinion poll (fictional) consists of the following three questions:

(1) Are you a registered Democrat?
(2) Do you approve of President Obama’s performance in office?
(3) Do you favor the Health Care Bill?

A group of 1000 people is polled. Answers to the questions are either yes or no. It is found that 550 people answer yes to the third question and 450 answer no. 325 people answer yes exactly twice (i.e. their answers contain 2 yeses and one no). 100 people answer yes to all three questions. 125 registered Democrats approve of Obama’s performance. How many of those who favor the Health Care Bill do not approve of Obama’s performance and in addition are not registered Democrats? (Hint: use a Venn diagram.)
Solution: Let $A$ denote the set of people who voted yes on question one, $B$ denote the set of people who voted yes on question two, and $C$ denote the set of people who voted yes on question 3.

We will denote the size of a set with bars $|·|$, so for example when the question tells us that “550 people answer yes to the third question and 450 answer no,” this becomes $|C| = 550$ and $|C^c| = 450$. With this notation, we can now translate rest of the paragraph:

(a) “325 people answer yes exactly twice” means that

$$|(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)| = 325$$

(b) “100 people answer yes to all three questions” means that

$$|A \cap B \cap C| = 100$$

(c) “125 registered Democrats approve of Obama’s performance” means that

$$|A \cap B| = 125.$$  

Notice that $A \cap B = (A \cap B \cap C) \cup (A \cap B \cap C^c)$.

(d) The question in the problem, "How many of those who favor the Health Care Bill do not approve of Obama’s performance and in addition are not registered Democrats?", is asking us to compute what $|A^c \cap B^c \cap C|$ is equal to.

At this point, we should try to fill out as many of the sizes of these sets as we can. For example, we know that

$$125 = |A \cap B| = |A \cap B \cap C| + |A \cap B \cap C^c| = 100 + |A \cap B \cap C^c|$$

and therefore $|A \cap B \cap C^c| = 25$. With this, we then know that

$$325 = |A \cap B \cap C^c| + |A \cap B^c \cap C| + |A^c \cap B \cap C|$$

$$= 25 + |A \cap B^c \cap C| + |A^c \cap B \cap C|$$

so we conclude that $|A \cap B^c \cap C| + |A^c \cap B \cap C| = 300$. But now, we know that $|C| = 550$ and that

$$550 = |C| = |A^c \cap B^c \cap C| + |A^c \cap B \cap C| + |A \cap B \cap C| + |A \cap B \cap C|,$$

$$= |A^c \cap B^c \cap C| + 300 + 100$$
and therefore $|A^c \cap B^c \cap C| = 150$.

6. We toss a coin twice. We consider three steps in this experiment. 1. before the first toss; 2. after the first toss but before the second toss; 3. after the two tosses.

a Give a sample space $\Omega$ for this experiment.

b Give the collection $\mathcal{F}_3$ of observable events at step 3.

c Give the collection $\mathcal{F}_2$ of observable events at step 2.

d Give the collection $\mathcal{F}_1$ of observable events at step 1.

**Solution:**

a $\Omega = \{(H_1, H_2), (H_1, T_2), (T_1, H_2), (T_1, T_2)\}$. Notationally, $(H_1, H_2)$ means "heads on the first flip, heads on the second" and similarly for the rest.

b At step 3, we have seen the result of the experiment, so $\mathcal{F}_3$ is the set of all subsets of $\Omega$, which is the usual choice of $\mathcal{F}$ that we deal with.

c At step 2, we have only observed the first toss. This means that the information in the second toss is not yet available, so we cannot distinguish for example between $(H_1, H_2)$ and $(H_1, T_2)$. This is because they only differ in the result of the second coin flip, which we have not yet observed. Once we know whether the first flip is heads or not, we also know whether it is tails or not. We see from this that $\mathcal{F}_2$ is the $\mathcal{F}$ that is generated by the event \{(H_1, H_2), (H_1, T_2)\}:

$$\mathcal{F}_2 = \{\emptyset, \{(H_1, H_2), (H_1, T_2)\}, \{(T_1, H_2), (T_1, T_2)\}, \{(H_1, H_2), (H_1, T_2), (T_1, H_2), (T_1, T_2)\}\}.$$ 

d At step one, nothing has happened yet so the only information available to us is that something will happen once we run the experiment. This corresponds to the choice

$$\mathcal{F}_1 = \{\emptyset, \{(H_1, H_2), (H_1, T_2), (T_1, H_2), (T_1, T_2)\}\}$$

7. A fair die is rolled 5 times and the sequence of scores recorded.

a How many outcomes are there?

b Find the probability that the first and last rolls are 6.

**Solution:**

a In each roll, there are 6 possibilities, so there are $6^5$ possible distinct combinations.

b There are $6^3$ possible sequences of scores which begin and end with 6 (this is the number of ways of choosing the remaining 3 rolls). Therefore the probability is $6^3/6^5 = 1/36$.

8. An urn contains 3 red, 8 yellow, and 13 green balls; another urn contains 5 red, 7 yellow, and 6 green balls. We pick one ball from each urn at random. Find the probability that both balls are of the same color.
Solution: There are $3 \times 5$ ways to pick two red balls, $8 \times 7$ ways to pick two yellow balls and $13 \times 6$ ways to pick two green balls. In total, this means that there are 149 ways to pick two balls of the same color. In total, there are $(3 + 8 + 13) \times (5 + 7 + 6) = 432$ ways to pick two balls from these two urns. Therefore the probability is $\frac{149}{432}$. 