Instructions

1. You will have 80 minutes to complete this exam. There are eight questions. Some questions are worth 10 points and some are worth 15 points. The points values of each question are marked next to the question.

2. Please write neatly! If I cannot read or understand what you write, I will assume that it is wrong.

3. You **DO NOT** need to simplify your answers. Do not spend time simplifying fractions.

4. You may have **one** 3 inch by 5 inch notecard with notes on it on your desk. You may have writing on both sides of the card.

5. You may not use any electronic devices (including calculators), books, or notes other than those on your **one** 3 inch by 5 inch index card.
1. (10 points) For each statement below, indicate whether the statement is true or false. If the statement is false, provide a counter-example.

(a) Suppose that $X$ is a discrete random variable and $g$ is a function on $\mathbb{R}$. Then $Y = g(X)$ is a discrete random variable.

(b) Suppose that $X$ is a continuous random variable and $g$ is a function on $\mathbb{R}$. Then $Y = g(X)$ is a continuous random variable.
2. *(10 points)* Suppose that we pick two numbers uniformly at random with replacement from the set \{2, 3\}. Let \(X\) be the minimum of the two numbers we choose and let \(Y\) be the maximum.

(a) Compute the joint probability mass function of \(X\) and \(Y\), \(f_{X,Y}(x, y)\).

(b) Find the marginal probability mass functions of \(X\) and \(Y\), \(f_X(x)\) and \(f_Y(y)\). Are \(X\) and \(Y\) independent?

(c) Compute \(E[XY]\).
3. (15 points) Suppose that $X$ is a random variable with probability density function

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$ 

Suppose that $Y = 1 - \log(1 + X)$. Find the probability density function $f_Y(y)$ and cumulative distribution function $F_Y(y)$ of $Y$. 
4. (15 points) Suppose that $U$ is a random variable which is uniformly distributed on the set $(0, 1)$ and $X$ is a random variable with probability density function

$$f_X(x) = \frac{1}{2}e^{-|x|}$$

Find a function $g$ so that $g(U)$ has the same distribution as $X$. 
5. (15 points) Suppose that $X$ is a random variable with probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

and let $Z = X^2$.

(a) Find the probability density function of $Z$, $f_Z(z)$.

(b) Compute $E[Z^{3/2}]$. 
6. (15 points) Suppose that X and Y are random variables with joint probability density function

\[ f_{X,Y}(x, y) = \begin{cases} C_{xy} & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases} \]

for some real number \( C > 0 \).

(a) Find C.

(b) Compute the marginal distributions of X and Y, \( f_X(x) \) and \( f_Y(y) \). Are X and Y independent?

(c) Compute \( P(X \leq Y^2) \).
7. (20 points) Suppose that \( X \) and \( Y \) are random variables with joint probability density function

\[
f_{X,Y}(x, y) = \begin{cases} 
    ye^{-y} & y \geq 0, 0 \leq x \leq 1 \\
    0 & \text{otherwise}
\end{cases}
\]

Let \((U, V) = (XY, (1 - X)Y)\).

(a) Show that the joint probability distribution of \((U, V)\) is

\[
f_{U,V}(u, v) = \begin{cases} 
    e^{-(u+v)} & u \geq 0, v \geq 0 \\
    0 & \text{otherwise}
\end{cases}
\]

(b) Are \( U \) and \( V \) independent? Justify your answer.

(c) Compute \( P(U \leq V) \).

(d) Compute \( E[UV] \).
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