Instructions

1. You will have 80 minutes to complete this exam. There are eight questions. Some questions are worth 10 points and some are worth 15 points. The points values of each question are marked next to the question.

2. Please write neatly! If I cannot read or understand what you write, I will assume that it is wrong.

3. You **DO NOT** need to simplify your answers. Do not spend time simplifying fractions.

4. You may have one 3 inch by 5 inch notecard with notes on it on your desk. You may have writing on both sides of the card.

5. You may not use any electronic devices (including calculators), books, or notes other than those on your one 3 inch by 5 inch index card.
1. (15 points) Suppose that we pick a number in \{1, 2\} uniformly at random and then toss that many fair coins.

(a) Write down a sample space \( \Omega \) and probabilities \( P \) corresponding to this situation.

(b) Let \( X \) denote the random variable which counts the number of heads we observe. Write down the probability mass function of \( X \).

(c) What is the cumulative distribution function of \( X \)?

Solution:

(a) \( \Omega = \{1H, 1T, 2HH, 2HT, 2TH, 2TT\}. P(1H) = P(1T) = 1/4, P(2HH) = P(2HT) = P(2TH) = P(2TT) = 1/8. 

(b) The possible values of \( X \) are \{0, 1, 2\}.

\[
f(k) = \begin{cases} 
P(1T) + P(2TT) & k = 0 \\
P(1H) + P(2HT) + P(2TH) & k = 1 \\
P(2HH) & k = 2 \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
\frac{3}{8} & k = 0 \\
\frac{1}{4} & k = 1 \\
\frac{1}{8} & k = 2 \\
0 & \text{otherwise}
\end{cases}
\]

(c)

\[
F(x) = \begin{cases} 
0 & x < 0 \\
\frac{3}{8} & 0 \leq x < 1 \\
\frac{7}{8} & 1 \leq x < 2 \\
1 & x \geq 2
\end{cases}
\]
2. (10 points) Suppose that you draw 5 cards randomly from a standard deck of 52 cards, without replacement. What is the probability of drawing a straight flush which is not a royal flush? A straight flush which is not a royal flush is a collection of 5 cards of the same suit and having consecutive ranks, where the lowest card is one of Ace, 2, 3, 4, 5, 6, 7, 8. There are 4 suits, each consisting of 13 ranks in a standard deck of cards. Explain how you arrived at your answer.

Solution: There are \( \binom{52}{5} \) possible hands of cards drawn from a standard deck without replacement. To draw a straight flush which is not a royal flush, we need to pick the suit from among the 4 suits of the flush and then the lowest card from among \{ A, 2, 3, 4, 5, 6, 7, 8 \}. There are 4 choices for the suit and 9 choices for the low card. The probability is then

\[
\frac{36}{\binom{52}{5}}.
\]
3. (10 points) Suppose that we have two urns. The first urn contains two red balls and one black ball. The second urn contains one red ball, one yellow ball, and one black ball.

(a) Suppose that we draw one ball uniformly at random from each urn. What is the probability that we draw exactly one red ball?

(b) Suppose that we pick one of the two urns uniformly at random and select two balls randomly from the urn, with replacement (this means that we draw a ball, then put it back into the same urn and pick another ball randomly from that urn). What is the probability that we draw two red balls?

Solution: We will denote drawing a red ball with an R, a yellow ball with a Y and a black ball with a B. Notice that the sample spaces are different for each part of the problem, because the experiments are all different.

(a) Denote the event that we draw a red ball on the first draw by $R_1$ and the event that we draw a black ball on the first draw by $B_1$. Similarly define $R_2$, $B_2$, $Y_2$ for the events of drawing a red, black, or yellow ball on the second draw. The draws here are independent. The event that we draw exactly one red ball is $(R_1 \cap B_2) \cup (R_1 \cap Y_2) \cup (B_1 \cap R_2)$

$$P((R_1 \cap B_2) \cup (R_1 \cap Y_2) \cup (B_1 \cap R_2)) = P(R_1 \cap B_2) + P(R_1 \cap Y_2) + P(B_1 \cap R_2)$$

$$= P(R_1)P(B_2) + P(R_1)P(Y_2) + P(B_1)P(R_2)$$

$$= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$

$$= \frac{5}{9}$$

(b) Let $U_1$ denote the event that the urn we selected was the first urn, $U_2$ denote the event that we selected the second urn, and let $R_1$ and $R_2$ be the events that we draw a red ball on the first draw and a red ball on the second draw respectively. Since $U_2 = U_1^c$, the law of total probability says that

$$P(R_1 \cap R_2) = P(R_1 \cap R_2|U_1)P(U_1) + P(R_1 \cap R_2|U_2)P(U_2).$$

Conditioned on selecting the first urn, there are 9 ways to draw two balls from the urn, with replacement if order matters. In 4 of these, we draw two red balls, therefore $P(R_1 \cap R_2|U_1) = 4/9$. Similarly there are 9 ways to draw two balls from urn two, but only in one of these do we draw two red balls. We see that

$$P(R_1 \cap R_2) = P(R_1 \cap R_2|U_1)P(U_1) + P(R_1 \cap R_2|U_2)P(U_2).$$

$$= \left(\frac{4}{9}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{9}\right)\left(\frac{1}{2}\right)$$

$$= \frac{5}{18}. $$
4. (10 points) Toss a fair coin three times. Let $A_1$ denote the event that the first and second tosses give the same outcome, let $A_2$ denote the event that the first and third tosses give the same outcome, and let $A_3$ denote the event that the second and third tosses give the same outcome.

1. Are $A_1$ and $A_2$ independent? Show your work.

2. Is the collection $\{A_1, A_2, A_3\}$ mutually independent? Show your work.

Solution: We can compute that

\[
P(A_1) = P(\text{HHH}) + P(\text{HHT}) + P(\text{TTT}) + P(\text{TTT}) = \frac{4}{8} = \frac{1}{2}
\]

\[
P(A_2) = P(\text{HHH}) + P(\text{HTH}) + P(\text{TTH}) + P(\text{TTT}) = \frac{4}{8} = \frac{1}{2}
\]

\[
P(A_3) = P(\text{HHH}) + P(\text{THH}) + P(\text{HTT}) + P(\text{TTT}) = \frac{4}{8} = \frac{1}{2}
\]

1. Yes. $P(A_1 \cap A_2) = P(\text{HHH}) + P(\text{TTT}) = \frac{1}{4} = P(A_1)P(A_2)$.

2. No. $P(A_1 \cap A_2 \cap A_3) = P(\text{HHH}) + P(\text{TTT}) = \frac{1}{4} \neq P(A_1)P(A_2)P(A_3)$. 
5. (15 points) Suppose that $A \subset B$, show that $P(B \cap A^c) = P(B) - P(A)$. Does this remain true in general if $A$ is not a subset of $B$? If not, provide a counterexample.

Solution: If $A \subset B$ then $B \cap A = A$ so

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(A) + P(B \cap A^c).$$

Therefore $P(B) - P(A) = P(B \cap A^c)$.

Almost any example of two events where one is not a subset of the other will work. For example, let $\Omega = \{HH, HT, TH, TT\}$ with $P(HH) = P(HT) = P(TH) = P(TT)$. Let $A = \{HH\}$ and $B = \{TT\}$. Then $A \cap B = \emptyset$, so $A$ is not a subset of $B$. $B \cap A^c = \{TT\} \cap \{HT, TH, TT\} = \{TT\} = B$. $P(B) - P(A) = 1/4 - 1/4 = 0$, but $P(B \cap A^c) = P(B) = 1/4$. 
6. (10 points) Suppose that $A$ and $B$ are events in $\Omega$ with $P(A), P(A^c), P(B) > 0$.

(a) Show that $P(A \cap B) = P(B|A)P(A)$ and $P(A^c \cap B) = P(B|A^c)P(A^c)$.

(b) Show that $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$.

(c) Show that

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}.$$ 

Justify your steps.

**Solution:**

(a) Since $P(A) > 0$,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$ 

Multiplying through by $P(A)$ gives $P(B|A)P(A) = P(A \cap B)$. Since $P(A^c) > 0$, $P(B|A^c)$ also makes sense and

$$P(B|A^c) = \frac{P(A^c \cap B)}{P(A^c)}.$$ 

Multiplying through by $P(A^c)$ gives $P(B|A^c)P(A^c) = P(A^c \cap B)$.

(b) Since $A \cup A^c = \Omega$ and $A$ and $A^c$ are pairwise disjoint, we have

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A)P(A) + P(B|A^c)P(A^c).$$

The last step is the first part of this problem.

(c) Since $P(B) > 0$, $P(A|B)$ makes sense. We have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}.$$

The last step uses part (a) of the problem in the numerator and part (b) in the denominator.
7. (15 points) Suppose that we have three coins on a table. The first coin is fair and flips heads with probability $1/2$. The second coin flips heads with probability $1/8$ and the third coin flips heads with probability $3/4$.

(a) Suppose that we select one coin of the three coins uniformly at random. What is the probability that it flips heads?

(b) Suppose that we select one coin uniformly at random, flip it, and the result of our flip is heads. What is the probability that the third coin was selected?

Solution:

(a) Denote by $C_1$ the event that we select the first coin, $C_2$ the event that we select the second coin, and $C_3$ the event that we select the third coin. Denote by $H$ the event that the result of our flip is a heads.

\[
P(H) = P(H|C_1)P(C_1) + P(H|C_2)P(C_2) + P(H|C_3)P(C_3)
\]

\[
= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{8}\right)\left(\frac{1}{3}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{3}\right)
\]

\[
= \frac{1}{6} + \frac{1}{24} + \frac{3}{12} = \frac{7}{24}
\]

(b) \[
P(C_3|H) = \frac{P(H|C_3)P(C_3)}{P(H|C_1)P(C_1) + P(H|C_2)P(C_2) + P(H|C_3)P(C_3)}
\]

\[
= \frac{\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)}{\frac{7}{24}} = \frac{6}{7}.
\]
8. \( (15 \text{ points}) \) Suppose that \( X \) is a random variable with cumulative distribution function

\[
F(x) = \begin{cases} 
0 & x < -1 \\
\frac{1}{4} & -1 \leq x < 0 \\
\frac{1}{4} + x^2 & 0 \leq x < \frac{1}{2} \\
\frac{1}{2} + \frac{1}{2}x & \frac{1}{2} \leq x < 1 \\
1 & x \geq 1
\end{cases}
\]

(a) Is this random variable continuous, discrete, or neither? Why?

(b) For which values of \( x \) is \( P(X = x) > 0 \)?

(c) Compute \( P(1/4 < X \leq 1/2) \).

Solution:

(a) This random variable is neither discrete nor continuous. We can see that it is not continuous because \( P(X = -1) = 1/4 - 0 = 1/4 \). We can see that it is not discrete because the possible values of \( X \) contain all values in \([0, 1)\) (this is because the CDF is continuous and strictly increasing on this region, which means that \( P(a < X \leq b) > 0 \) for any \( a \) and \( b \) with \( 0 \leq a < b < 1 \)).

(b) \( P(X = x) > 0 \) for \( x = -1 \) and \( x = 1/2 \), since these are the only values of \( x \) for which \( F(x) - \lim_{y \uparrow x} F(y) > 0 \) (notice that the left and right limits match at \( x = 0 \) and \( x = 1 \)).

(c) \( P(1/4 < X \leq 1/2) = F(1/2) - F(1/4) = 1/2 + 1/4 - (1/4 + 1/16) = 7/16. \)