Instructions

1. You will have 80 minutes to complete this exam. There are eight questions. Some questions are worth 10 points and some are worth 15 points. The points values of each question are marked next to the question.

2. Please write neatly! If I cannot read or understand what you write, I will assume that it is wrong.

3. You **DO NOT** need to simplify your answers. Do not spend time simplifying fractions.

4. You may have one 3 inch by 5 inch notecard with notes on it on your desk. You may have writing on both sides of the card.

5. You may not use any electronic devices (including calculators), books, or notes other than those on your one 3 inch by 5 inch index card.
1. (15 points) Suppose that we pick a number in \{1, 2\} uniformly at random and then toss that many fair coins.

(a) Write down a sample space \( \Omega \) and probabilities \( P \) corresponding to this situation.

(b) Let \( X \) denote the random variable which counts the number of heads we observe. Write down the probability mass function of \( X \).

(c) What is the cumulative distribution function of \( X \)?
2. (10 points) Suppose that you draw 5 cards randomly from a standard deck of 52 cards, without replacement. What is the probability of drawing a straight flush which is not a royal flush? A straight flush which is not a royal flush is a collection of 5 cards of the same suit and having consecutive ranks, where the lowest card is one of Ace, 2, 3, 4, 5, 6, 7, 8, 9. There are 4 suits, each consisting of 13 ranks in a standard deck of cards. Explain how you arrived at your answer.
3. (10 points) Suppose that we have two urns. The first urn contains two red balls and one black ball. The second urn contains one red ball, one yellow ball, and one black ball.

(a) Suppose that we draw one ball uniformly at random from each urn. What is the probability that we draw exactly one red ball?

(b) Suppose that we pick one of the two urns uniformly at random and select two balls randomly from the urn, with replacement (this means that we draw a ball, then put it back into the same urn and pick another ball randomly from that urn). What is the probability that we draw two red balls?
4. (10 points) Toss a fair coin three times. Let $A_1$ denote the event that the first and second tosses give the same outcome, let $A_2$ denote the event that the first and third tosses give the same outcome, and let $A_3$ denote the event that the second and third tosses give the same outcome.

1. Are $A_1$ and $A_2$ independent? Show your work.

2. Is the collection $\{A_1, A_2, A_3\}$ mutually independent? Show your work.
5. (15 points) Suppose that \( A \subset B \), show that \( P(B \cap A^c) = P(B) - P(A) \). Does this remain true in general if \( A \) is not a subset of \( B \)? If not, provide a counterexample.
6. (10 points) Suppose that $A$ and $B$ are events in $\Omega$ with $P(A), P(A^c), P(B) > 0$.

(a) Show that $P(A \cap B) = P(B|A)P(A)$ and $P(A^c \cap B) = P(B|A^c)P(A^c)$.

(b) Show that $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$.

(c) Show that

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}.$$

Justify your steps.
7. (15 points) Suppose that we have three coins on a table. The first coin is fair and flips heads with probability 1/2. The second coin flips heads with probability 1/8 and the third coin flips heads with probability 3/4.

(a) Suppose that we select one coin of the three coins uniformly at random. What is the probability that it flips heads?

(b) Suppose that we select one coin uniformly at random, flip it, and the result of our flip is heads. What is the probability that the third coin was selected?
8. \textit{(15 points)} Suppose that \( X \) is a random variable with cumulative distribution function

\[
F(x) = \begin{cases} 
0 & x < -1 \\
\frac{1}{4} & -1 \leq x < 0 \\
\frac{1}{4} + x^2 & 0 \leq x < \frac{1}{2} \\
\frac{1}{2} + \frac{1}{2}x & \frac{1}{2} \leq x < 1 \\
1 & x \geq 1
\end{cases}
\]

(a) Is this random variable continuous, discrete, or neither? Why?

(b) For which values of \( x \) is \( P(X = x) > 0? \)

(c) Compute \( P(1/4 < X \leq 1/2). \)