Math 6320, Assignment 4 **Due: March 23rd**

1. Determine the minimal polynomial of $\cos 2\pi/7$ over $\mathbb{Q}$.

2. If n is an odd positive integer, prove that $\mathbb{Q}(\cos(2\pi/n)) = \mathbb{Q}(\cos(\pi/n))$.

3. Take a regular n-sided polygon inscribed in a circle of radius 1. Label the vertices $P_1, \ldots, P_n$, and let λ_k be the length of the line joining P_k and P_n for $1 \leq k \leq n-1$. Prove that

$$\lambda_1 \cdots \lambda_{n-1} = n\,.$$

4. Find a real number α such that the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$ is Galois, with Galois group $\mathbb{Z}/6$.

5. Let α be the positive real number $5^{1/4}$. Prove that each of the extensions $\mathbb{Q} \subset \mathbb{Q}(i\alpha^2)$ and $\mathbb{Q}(i\alpha^2) \subset \mathbb{Q}(\alpha + i\alpha)$ is normal. Is $\mathbb{Q} \subset \mathbb{Q}(\alpha + i\alpha)$ normal?

6. Let $E = \mathbb{Q}(e^{2\pi i/8})$. Determine all subgroups of $\mathrm{Gal}(E/\mathbb{Q})$, and the corresponding intermediate fields.

7. Let $\underline{s} := s_1, s_2, s_3$ be the elementary symmetric polynomials in $\underline{x} := x_1, x_2, x_3$. Prove that $\mathbb{Q}(\underline{x})$ is not a radical extension of $\mathbb{Q}(\underline{s})$, and that it is a radical extension of $\mathbb{Q}(i, \underline{s})$.

8. Prove that $x^7 - 10x^5 + 15x + 5$ is solvable by radicals over $\mathbb{Q}$.

9. Let $f(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$ of prime degree $p \geq 5$, and E the splitting field of $f(x)$. Prove that $f(x)$ is solvable by radicals if and only if $E = \mathbb{Q}(\alpha, \beta)$ for any pair of distinct roots α, β of $f(x)$.

10. Deduce from the preceding exercise that if $f(x)$ as above has exactly two real roots, then it is not solvable by radicals. (This was proved by Galois).