Math 6310, Assignment 5 **Due: 9th November, Monday**

1. If $1-ab$ is invertible in a ring, show that $1-ba$ is also invertible.
2. Let R be a ring in which $x^3 = x$ for each x. Prove that R is commutative.

In the problems below, R is a commutative ring.

3. In which of the following rings is every ideal principal? Justify your answer.

$$\text{(a) } \mathbb{Z}\times\mathbb{Z}, \qquad \text{(b) } \mathbb{Z}/4, \qquad \text{(c) } (\mathbb{Z}/6)[x], \qquad \text{(d) } (\mathbb{Z}/4)[x].$$

4. If R is a domain that is not a field, prove that the polynomial ring $R[x]$ is not a principal ideal domain.
5. An element r in a ring R is *nilpotent* if $r^n = 0$ for some $n \geqslant 0$. Prove the following assertions.
   (a) If r is nilpotent, then $1+r$ is invertible in R.
   (b) If $r_1,\ldots,r_c$ are nilpotent elements, then any element in the ideal $(r_1,\ldots,r_c)$ is nilpotent.
6. Let R be a commutative ring and $R[x]$ the polynomial ring over R in the indeterminate x. Let

$$f(x) = r_0 + r_1x + r_2x^2 + \cdots + r_nx^n \quad \text{with } r_i \in R.$$

Prove the following assertions.

   (a) $f(x)$ is nilpotent if and only if $r_0,\ldots,r_n$ are nilpotent.
   (b) $f(x)$ is a unit in $R[x]$ if and only if r_0 is a unit in R and $r_1,\ldots,r_n$ are nilpotent.
   (c) $f(x)$ is a zerodivisor if and only if there exists a nonzero element $r \in R$ such that $r \cdot f = 0$.

Recall that $a \in R$ is a *zerodivisor* if there exists $b \neq 0$ in R with $ab = 0$; the only zerodivisor in a domain is 0.