Math 6310, Assignment 1 **Due in class: Friday, September 4**

1. Suppose G is a finite set with an associative law of composition, and $e \in G$ is an element such that $xe = x = ex$ for all $x \in G$. If G has the property that
$$xz = yz \quad \text{implies} \quad x = y,$$
prove that G is a group.

2. Let G be a group, and let H be a subgroup of finite index. Prove that the number of right cosets of H equals the number of left cosets.

3. Let a be an element of a group G. Prove that there exists x in G with $x^2ax = a^{-1}$ if and only if a is the cube of some element of G.

4. Let G be a finite group. If $x^2 = e$ for each $x \in G$, prove that $|G|$ is a power of 2.

5. Find a group with elements a, b such that a and b have finite order, but ab does not have finite order. (Hint: Try looking in $\mathrm{GL}_2(\mathbb{Z})$, the group of invertible 2×2 matrices over $\mathbb{Z}$.)

6. Let n be a positive integer. Consider the set G of positive integers less than or equal to n that are relatively prime to n. The number of elements of G is the *Euler phi-function*, denoted $\phi(n)$.

   (a) Show that G is a group under multiplication modulo n.

   (b) If m and n are relatively prime positive integers, show that $m^{\phi(n)} \equiv 1 \bmod n$.

7. Let n be a positive integer. Show that $n = \sum_{d|n} \phi(d)$, where the sum is taken over all positive integers d that divide n. (Hint: A cyclic group of order n has a unique subgroup of order d for each d dividing n.)

8. Let G be a finite group with the property that for each integer $d \geq 1$, the equation $x^d = e$ has at most d solutions in G. Prove that G is cyclic.

9. Let G be a group such that for a fixed integer $n > 1$, we have $(xy)^n = x^n y^n$ for all $x, y \in G$. Let
$$G^{(n)} = \{x^n \mid x \in G\} \quad \text{and} \quad G_{(n)} = \{x \in G \mid x^n = e\}.$$

   (a) Prove that $G^{(n)}$ and $G_{(n)}$ are normal subgroups of G.

   (b) If G is finite, show that the order of $G^{(n)}$ equals the index of $G_{(n)}$.

   (c) Show that for all $x, y \in G$, we have $x^{1-n}y^{1-n} = (xy)^{1-n}$. Use this to get $x^{n-1}y^n = y^n x^{n-1}$.

   (d) Conclude that elements of G of the form $x^{n(n-1)}$ generate an abelian subgroup.

10. Let G be a group such that $(xy)^3 = x^3y^3$ for all $x, y \in G$, and such that the map $x \mapsto x^3$ is bijective. Prove that the group G is abelian.