POLYMATH JR. 2025 - HOWE PROJECT DESCRIPTION

1. Summary

The goal of this project is to compute the "Λ-distributions" of natural matrix-valued random variables and then to derive some interesting consequences of these computations. Let me explain in rough terms what this means:

Suppose I have a way to pick an $n \times n$ matrix M at random. To understand the properties of this random process, I might try to associate some numbers to it, and then ask how they are distributed. For example, I might ask: what is the average value of the trace of my randomly chosen M?

The trace of a matrix is most easily defined as the sum of the diagonal entries, but, it is also equal to the sum of its eigenvalues. More generally, one could ask for the average value of *any* polynomial of the eigenvalues. Since the eigenvalues (counted with multiplicity) do not have a natural order, this only makes sense if we use a symmetric polynomial, i.e. a polynomial $f(x_1, \ldots, x_n)$ such that $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for any bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

The Λ-distribution is a way of encoding all of these averages; by the theory of moments, it also encodes the classical distributions (in the sense of probability theory) of any of the individual quantities such as the trace.

By some small miracle, the Λ-distribution often has a very simple description via its "σ-moment generating function," which is some nice way of putting all of these numbers together as the coefficients of a power series in infinitely many variables. For example, one finds that random orthogonal matrices (matrices whose transpose is equal to their own inverse), have a "Λ-gaussian" distribution encoded by the σ-moment generating function

$$\mathrm{Exp}_\sigma(h_2),$$

where Exp_σ is some funny version of the usual exponential e^x and

$$h_2(t_1, t_2, t_3, \ldots) = (t_1^2 + t_1 t_2 + t_1 t_3 + \ldots) + (t_2^2 + t_2 t_3 + \ldots) + \ldots$$

is the formal sum of all degree two monomials in infinitely many variables $t_1, t_2, \ldots$. This might look complicated at first, but an equivalent result was described earlier by Diaconis and Shahshahani using formulas that take almost a page to write down completely! Not so bad now, right?

These kinds of succint descriptions have nice applications, e.g., in number theory. But we probably won't get into that — mostly we want to compute new examples.

2. Goals

In [H1], I computed the asymptotic Λ-distributions of some random matrices in important families (standard representations of the "classical" compact groups). There are many more interesting cases to consider, especially for finite matrix groups (for example, the group of even permutation matrices).

The first goal will thus be to write some computer programs that first check the earlier results for classical groups as a proof-of-concept, then compile data in many new cases. This mostly involves linear algebra, elementary probability theory, and some basic programming.

The second goal is then to take this data and try to find simple descriptions of the σ-moment generating functions. This involves learning a decent amount of the theory of symmetric functions, but from a very computational perspective. There is a lot of computer software that can be used to help with this as well.

The third goal is to analyze the data both in the cases already treated and in new cases to see if one can improve asymptotics for the convergence of certain distributions. This is the most difficult part of the project, and it will involve some non-trivial real analysis and a more refined understanding of moments. On the other hand, the first step will simply be to identify some patterns, so it should be possible to make contributions at many levels.

Along the way, there are many natural points for the project to branch off into different directions, according to your interests! I will highlight some of these after we get started. In the end, we will produce a website/software that makes it easy to look up the $\Lambda-$distributions for many matrix representations of interesting groups, showcasing all of our results from part 1. If we are successful at all in the second or third part, we will write a paper too. Λ-distributions and σ-moment generating functions are a very new tool, so there are lots of interesting things to do with it, and many are accessible without much background!

3. Prerequisites and learning resources

The only real pre-requisite is a solid grasp of linear algebra, especially eigenvalues and characteristic polynomials — if you want to work on this project, make sure you review these ideas first! A basic command of finite groups and the following topics will also be useful, but will be covered somewhat in the first week or two:

- Some probability theory (it will suffice largely to understand finite probability spaces with the uniform measure!). Key concepts will be: independence, the weak law of large numbers / Monte Carlo methods, moments, and moment generating functions. Possible references:
  - A nice introductory text is Grimmett and Welsh's Probability: An Introduction
  - A nice more advanced book is Billingsley — Probability and Measure.
- Some basic theory of symmetric functions. Possible references:
  - Many undergraduate abstract algebra books will have a some discussion of symmetric functions. Check yours if you have one!
  - Symmetric Functions: A Beginner's Course by Tutubalina and Smirnov
- Some representation theory of finite groups / matrix groups. (What we actually need is much less than is found in the following). Possible references:
  - Many undergraduate abstract algebra books will have a some discussion of the representation theory of finite groups. Check yours if you have one!
  - Tapp — Matrix Groups for Undergraduates
  - Fulton-Harris — Representation Theory: A First Course (just the first chapter, which also has symmetric functions!)
  - Serre — Linear Representations of Finite Groups (just the first half)
- We will be doing a lot of computations/programming, but most will not be too complicated. I like Sage/Python (you can use it online for quick

computations at `https://sagecell.sagemath.org` or more robustly at `cocalc.org`), which has, in particular, good symmetric function support. If you've never done any programming you might take a look at a Sage tutorial (`https://doc.sagemath.org/html/en/tutorial/`).

4. Logistics

Mentor: Sean Howe
TAs: Matthew Bertucci, Will Dudarov, Matthew Hase-Liu

The main forum for interaction will be a persistent gather.town room (gather.town is like zoom but with a spatial aspect so you can walk away from a conversation/form groups/gather around a whiteboard, etc.) I'll plan to spend an hour or two there most days (with a semi-regular schedule), and we'll have one or two "official" recorded meetings each week. The room will always be open for you to meet with other group members, and TAs will also likely hold some regular office hours there. The first week of the project we will also have some crash courses on some key concepts for the project such as: eigenvalues, probability, matrix groups, symmetric functions, and matrix representations of groups, and then we'll figure out the structure from there based on what is working for people!

5. Bibliography

These are the sources for the basic definitions, but they will probably be hard to read at first unless you already have a lot of background in related material and experience reading research mathematics. We'll make the ideas more accessible once we get started in the project group!

(H1) Random matrix statistics and zeroes of L-functions via probability in pre-λ rings. `https://arxiv.org/pdf/2412.19295`. *If you look at this, stick with the intro through 1.2, then sections 2-4; don't worry about the number theory applications.*

(H2) The negative σ-moment generating function. `https://arxiv.org/abs/2505.01205`. *This is a supplement to (H1) that explains how to do one more important type of computation that might come up at some point.*