A UNIPOTENT CIRCLE ACTION ON p-ADIC MODULAR FORMS

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ABSTRACT. Following a suggestion of Peter Scholze, we construct an action of $\widehat{\mathbb{G}_m}$ on the Katz moduli problem over the ordinary locus of the modular curve at full prime-to-p level. This action is a local, p-adic analog of a global, archimedean action of S^1 on the unstable locus for modular curves over $\mathbb{C}$. We explain how the $\widehat{\mathbb{G}_m}$-action interacts with classical notions in the study of p-adic modular forms and modular curves, and use it to give a new representation-theoretic interpretation of Hida's ordinary p-adic modular forms analogous to the classical automorphic study of Eisenstein series through the constant term. Along the way, we also prove a natural generalization of Dwork's equation $\tau = \log q$ for extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\widehat{\mathbb{G}_m}$ valid over a non-Artinian base.

CONTENTS

1. INTRODUCTION

In this work, following a suggestion of Peter Scholze, we descend the unipotent quasi-isogeny action on a component of the ordinary (big) Igusa formal scheme of Caraiani-Scholze [1, Section 4] to construct an action of the formal p-adic torus $\widehat{\mathbb{G}_m}$ on the Katz moduli problem over the ordinary locus. Suitably interpreted, this action is a local, p-adic analog of the global, archimedean phenomena whereby the horizontal translation action of $\mathbb{R}$ on the complex upper half plane $\mathbb{H}$ descends to an action of S^1 on the image of $\{\mathrm{Im} z > 1\} \subset \mathbb{H}$ in the complex modular curve. The ring of functions on the Katz moduli problem is the space of p-adic modular forms, thus we may think of our $\widehat{\mathbb{G}_m}$-action as a unipotent circle action on p-adic modular forms. The analogy is stronger than one might first guess, and leads, e.g., to interesting representation-theoretic consequences.

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After constructing the $\widehat{\mathbb{G}_m}$-action, we study its properties and interaction with other classical notions in the p-adic theory of modular curves and modular forms such as the unit root splitting, the differential operator θ, Gouvea's [4] twisting measure, Dwork's equation $\tau = \log q$, Katz's Eisenstein measures, and ordinary p-adic modular forms à la Hida.

1.1. **An archimedean circle action.** Before stating our results, we explain the analogous archimedean circle action more carefully; this will help to motivate and clarify the p-adic constructions that follow. Consider the complex manifold

$$Y_{\infty-\mathrm{ord}} := \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \backslash \{\mathrm{Im}\tau > 1\}. \tag{1.1.0.1}$$

Two important observations about $Y_{\infty-\mathrm{ord}}$ follow immediately from (1.1.0.1):

(1) Modular forms of level $\Gamma_1(N)$ (for any N) restrict to $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$-invariant functions $\{\mathrm{Im} z > 1\}$, and thus induce holomorphic functions on Y_{ord}.
(2) The action of $\mathbb{R}$ by horizontal translation on $\mathbb{H}$ descends to a (real analytic) action of the circle group S^1 on Y_{ord}. This action integrates the vector field dual to $d\tau$.

We can decompose any holomorphic function f on $Y_{\infty-\mathrm{ord}}$ according to this S^1 action uniquely as a Fourier series

$$f(q) = \sum_{n\in\mathbb{Z}} a_n q^n,\ q = e^{2\pi i z}.$$

In other words, the space of functions on $Y_{\infty-\mathrm{ord}}$ is a Frechet completion of the direct sum of the character spaces for the S^1-action, with each character appearing exactly once.

1.1.1. *Fourier coefficients and representation theory.* The Fourier coefficients a_n of classical modular forms play an important role in the global automorphic representation theory for GL_2: for example, for a Hecke eigenform, the constant coefficient a_0 is non-vanishing if and only if the corresponding global automorphic representation is global principal series (i.e. the modular form is Eisenstein). Suitably interpreted, the constant term a_0 is a functional that realizes the induction. The non-constant coefficients, on the other hand, are Whittaker functionals.

1.1.2. *The slope formalism on metrized tori.* While the construction of $Y_{\infty-\mathrm{ord}}$ above may at first seem ad hoc, it has a natural moduli interpretation, which we explain now. The key point is to use the slope formalism for metrized tori, or, equivalently, lattices, as studied, e.g., in Casselman's survey [2].

A metrized torus is a finite dimensional torus (compact real abelian Lie group) T together with a translation invariant metric, or, equivalently, a positive definite inner product on $\mathrm{Lie}T \cong H_1(T, \mathbb{R})$. There is a natural slope formalism on metrized tori: the rank function is dimension, and the degree function is given by

$$\deg T := \log\mathrm{Vol}(T).$$

If a two-dimensional metrized torus T is unstable (i.e., not semi-stable), then it contains a unique circle of shortest length.

If $E/\mathbb{C}$ is an elliptic curve, the underlying real manifold of $E(\mathbb{C})$ is a two-dimensional metrized torus when equipped with the metric coming from the canonical principal polarization.

Example 1.1.3. Consider τ in the usual fundamental domain for the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H}$ (i.e., $-1/2 \leq \mathrm{Re}\tau \leq 1/2$, $|\tau| \geq 1$), and let

$$E_\tau = \mathbb{C}/\langle 1, \tau \rangle.$$

We compute the values of τ for which E_τ is semistable: for $\tau \in \mathbb{H}$, the metric induced by the principal polarization is identified with $1/\mathrm{Im}\tau$ times the metric induced by the identity

$$\mathbb{R}1 + \mathbb{R}\tau = \mathbb{C}$$

and the standard metric on $\mathbb{C}$. Semistability is preserved by scaling the metric, so we may eliminate the scaling and consider just the metric induced by the standard metric on $\mathbb{C}$. The length of a shortest circle in $E_\tau(\mathbb{C})$ is equal to the length of a shortest vector in $H_1(E_\tau, \mathbb{Z})$, which is 1. The area of the entire torus $E_\tau(\mathbb{C})$, on the other hand, is $\mathrm{Im}\tau$. Thus, the slope of the full torus is $\frac{1}{2}\log \mathrm{Im}\tau$, while the smallest slope of a circle inside is 0. We conclude that E_τ is semi-stable when $\mathrm{Im}\tau \leq 1$ (recalling that τ was assumed to be in the fundamental domain), and otherwise is unstable with shortest circle given by

$$S^1 = \mathbb{R}/\mathbb{Z} \hookrightarrow \mathbb{C}/\langle 1, \tau \rangle.$$

In particular, we may consider the moduli space of unstable elliptic curves $E/\mathbb{C}$ equipped with a trivialization of the shortest circle, $S^1 \hookrightarrow E(\mathbb{C})$. From Example 1.1.3, we find that this space is naturally identified with $Y_{\infty-\mathrm{ord}}$. In this moduli interpretation, the space $\mathrm{Im}\tau > 1$ is the cover where the trivialization of the shortest circle is extended to an oriented trivialization $S^1 \times S^1 \xrightarrow{\sim} E(\mathbb{C})$. From the moduli perspective, the fact that we can evaluate modular forms to obtain functions on $Y_{\infty-\mathrm{ord}}$ comes from two facts:

(1) Given a point of $Y_{\infty-\mathrm{ord}}$, there is a unique holomorphic differential ω_{can} whose pullback to S^1 along the trivialization of the shortest circle integrates to 1. Thus, the modular sheaf ω is canonically trivialized over $Y_{\infty-\mathrm{ord}}$, and modular forms can be evaluated along this trivialization,.

(2) Using the polarization, the trivialization of the shortest circle also gives rise to a trivialization of the quotient torus $E(\mathbb{C})/S^1$, so that $E(\mathbb{C})$ is equipped with the structure of an extension of real tori

$$1 \to S^1 \to E(\mathbb{C}) \to S^1 \to 1 \tag{1.1.3.1}$$

The basis $1/N$ for the torsion on $S^1 = \mathbb{R}/\mathbb{Z}$ then gives rise to a canonical $\Gamma_1(N)$-level structure on E for any level N.

1.1.4. *de Rham cohomology.* Consider the extension structure 1.1.3.1 on the universal elliptic curve over $Y_{\infty-\mathrm{ord}}$. The global section dx of the de Rham cohomology of S^1 is flat, so we obtain via pullback of dx a canonical flat section $u_{\mathrm{can}} \in H^1_{\mathrm{dR}}(E(\mathbb{C}), \mathbb{R})$ over $Y_{\infty-\mathrm{ord}}$. Moreover, because the image of ω_{can} in $H^1_{\mathrm{dR}}(S^1, \mathbb{C})$ under pullback is (by definition) dx, which is flat, we find that $\nabla(\omega_{\mathrm{can}})$ is in the span of u_{can}. Thus, we obtain a holomorphic differential form $\frac{\nabla \omega_{\mathrm{can}}}{u}$ on $Y_{\infty-\mathrm{ord}}$.

For the elliptic curve E_τ, if we denote by e_1 and e_τ the natural basis elements for $H_1(E(\mathbb{C}), \mathbb{Z})$ and by e_1^* and e_τ^* the dual basis, we find that $\omega_{\mathrm{can}} = e_1^* + \tau e_\tau^*$, and $u_{\mathrm{can}} = e_\tau^*$, so that

$$\frac{\nabla \omega_{\mathrm{can}}}{u} = d\tau = d \log q.$$

In particular, the S^1 action integrates the vector field dual to $\frac{\nabla\omega_{\mathrm{can}}}{u}$.

1.2. **Statement of results.** In this section we state our main results.

1.2.1. *Dictionary.* As we introduce the objects appearing in the local, p-adic theory, it may be helpful to keep in mind the following dictionary for our analogy with the global, archimedean story:

Global, archimedean	*Local, p-adic*
$E(\mathbb{C})$ as a metrized torus	The p-divisible group $E[p^\infty]$
Unstable two-dimensional metrized torus	Ordinary height two p-divisible group
The shortest circle in $E(\mathbb{C})$	The formal group $\widehat{E}$
Trivialization of the shortest circle $S^1 \hookrightarrow E(\mathbb{C})$	Trivialization of the formal group $\widehat{E} \xrightarrow{\sim} \widehat{\mathbb{G}_m}$
$1 \to S^1 \to E(\mathbb{C}) \to S^1 \to 1$	$1 \to \widehat{\mathbb{G}_m} \to E[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 1.$
$d\tau$, ω_{can}, u_{can}	$d\tau$, ω_{can}, u_{can}
Canonical $\Gamma_1(N)$ level structure	Canonical arithmetic $\Gamma_1(p^n)$ level structure
$Y_{\infty-\mathrm{ord}}$	The Katz moduli problem M_{Katz}
$\mathrm{Im}z \geq 1$	Connected component of $M_{\mathrm{big\,Igusa}}$
Action of S^1 on $Y_{\infty-\mathrm{ord}}$	Action of $\widehat{\mathbb{G}_m}$ on M_{Katz}
Action of $\mathbb{R}$ on $\mathrm{Im}z \geq 1$	Action of the universal cover on a connected component of $M_{\mathrm{big\,Igusa}}$
Fourier series	Sheaf over $\mathbb{Z}_p$
Constant term	Fiber at 0
Eisenstein series	Ordinary p-adic modular form

1.2.2. *The Katz moduli problem and p-adic modular forms.* Let R be a p-adically complete ring, and let Nilp_R be the category of R-algebras in which p is nilpotent. We consider the moduli problem $M_{\mathrm{Katz},R}$ on $\mathrm{Nilp}_R^{\mathrm{op}}$ classifying triples

$$(E, \widehat{\varphi}, \alpha)$$

where $E/\mathrm{Spec}R$ is an elliptic curve up to prime-to-p isogeny, $\widehat{\varphi}$ is a trivialization of the formal group of E,

$$\widehat{\varphi} : \widehat{E} \xrightarrow{\sim} \widehat{\mathbb{G}_m},$$

and α is a trivialization of the adelic prime-to-p Tate module.

By work of Katz [7], the moduli problem $M_{\mathrm{Katz},R}$ is represented by a p-adically complete ring $\mathbb{V}_{\mathrm{Katz},R}$, flat over R, and

$$\mathbb{V}_{\mathrm{Katz},R} = \mathbb{V}_{\mathrm{Katz}} \widehat{\otimes}_{\mathbb{Z}_p} R$$

where we denote $\mathbb{V}_{\mathrm{Katz}} := \mathbb{V}_{\mathrm{Katz},\mathbb{Z}_p}$.

The moduli problem M_{Katz} admits a natural action of $\mathbb{Z}_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$, where $\mathbb{Z}_p^\times = \mathrm{Aut}(\widehat{\mathbb{G}_m})$ acts by composition with $\widehat{\varphi}$, and $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$ acts by composition with α. For a continuous character κ of $\mathbb{Z}_p^\times$ with values in R, the eigenspace $\mathbb{V}_{\mathrm{Katz},R}[\kappa]$ is a natural space of p-adic modular forms of weight κ; in particular, classical modular forms of integral weight and prime-to-p level are embedded $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariantly (up to a twist) in this space for the character $z \mapsto z^k$. The embedding is given by evaluation on the trivialization of the modular sheaf ω given by the canonical differential $\omega_{\mathrm{can}} = \widehat{\varphi}^* \frac{dt}{t}$.

1.2.3. *de Rham cohomology.* Over M_{Katz}, we have the relative de Rham cohomology of the universal elliptic curve up-to-prime-to-p-isogeny

$$\pi : E_{\mathrm{univ}} \to M_{\mathrm{Katz}},$$

i.e. the vector bundle

$$H^1_{\mathrm{dR}}(E_{\mathrm{univ}}) := R^1\pi_*\Omega^\bullet_{E_{\mathrm{univ}}/M_{\mathrm{Katz}}}$$

equipped with its Hodge filtration

$$0 \to \omega_E \to H^1_{\mathrm{dR}} \to \mathrm{Lie}E^\vee \to 0$$

and Gauss-Manin connection ∇.

Note that the moduli problem M_{Katz} is equivalent to the moduli problem classifying triples $(E, \widehat{\varphi}, \alpha)$ where E and $\widehat{\varphi}$ are before, and α is a trivialization of the prime-to-p Tate module

$$\alpha : (\widehat{\mathbb{Z}}^{(p)})^2 \xrightarrow{\sim} T_{\widehat{\mathbb{Z}}^{(p)}}E = \lim_{(n,p)=1} E[n],$$

all considered up to *isomorphism* of E. Using this equivalence, we obtain a well-defined Weil pairing on $E[p^\infty]$, and combining this with the trivialization $\widehat{\varphi}$, we obtain the structure of an extension

$$1 \to \widehat{\mathbb{G}_m} \to E[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 1. \tag{1.2.3.1}$$

This is analogous to the archimedean extension (1.1.3.1); in particular, pulling back via the map $E[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p$ and using the the crystalline-de Rham comparison, we obtain a canonical flat section u_{can} in $H^1_{\mathrm{dR}}(E_{\mathrm{univ}})$ (spanning the unit-root filtration). Together ω_{can} and u_{can} are a basis for $H^1_{\mathrm{dR}}(E_{\mathrm{univ}})$, and in this basis ∇ is lower nilpotent and thus is determined by a single differential form

$$d\tau := \frac{\nabla(\omega_{\mathrm{can}})}{u_{\mathrm{can}}}.$$

By the theory of Kodaira-Spencer, the differential form $d\tau$ is non-vanishing, and thus admits a dual vector field $\frac{d}{d\tau}$ such that $\langle d\tau, \frac{d}{d\tau}\rangle = 1$.

1.2.4. *The $\widehat{\mathbb{G}_m}$ action.* Our main result, Theorem A below, shows that the vector field $\frac{d}{d\tau}$ can be integrated to an action of $\widehat{\mathbb{G}_m}$ on M_{Katz}, and explains how this action interacts with the action of $\mathbb{Z}_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$. To state it, we will need the unramified determinant character $\det_{\mathrm{ur}} : \mathrm{GL}_2(\mathbb{A}_f^{(p)}) \to \mathbb{Z}_{(p)}$ defined by

$$\det_{\mathrm{ur}}((g_l)_{l\neq p}) = \prod_{l\neq p} |\mathrm{det} g_l|_l.$$

Theorem A. *There is an action of $\widehat{\mathbb{G}_m}$ on M_{Katz} whose derivative, a vector field t_{u} on M_{Katz}, satisfies*

$$d\tau(t_{\mathrm{u}}) = 1. \tag{1.2.4.1}$$

Moreover, this combines with the action of $\mathbb{Z}_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ to give an action of

$$\widehat{\mathbb{G}_m} \rtimes (\mathbb{Z}_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)}))$$

where the semi-direct product is formed with the respect to the conjugation action

$$(z,g)\cdot\zeta\cdot(z,g)^{-1}=\zeta^{z^2\det_{\mathrm{ur}}(g)}.$$

The $\widehat{\mathbb{G}_m}$-action is uniquely determined by (1.2.4.1), which is another incarnation of the famous equation of Dwork, $\tau=\log q$. Moreover, (1.2.4.1) can be reformulated by saying that the $\widehat{\mathbb{G}_m}$-action integrates the derivation $-\theta$ which acts as $-q\frac{d}{dq}$ on cuspidal q-expansions.

The key observation in the construction of this $\widehat{\mathbb{G}_m}$-action action and subsequent computations is that we may work on a very ramified cover, a component of the (big) Igusa variety of Caraiani-Scholze [1, Section 4], where the extension structure (1.2.3.1) extends to a trivialization of the p-divisible group

$$\varphi: E[p^\infty]\xrightarrow{\sim}\widehat{\mathbb{G}}_m\times\mathbb{Q}_p/\mathbb{Z}_p.$$

At the price of the ramification, life is simplified on this cover: for example, computations with the crystalline connection are reduced to computing the crystalline realization of maps $\mathbb{Q}_p/\mathbb{Z}_p\to\widehat{\mathbb{G}_m}$. Most importantly, the obvious action of automorphisms of $\widehat{\mathbb{G}_m}\times\mathbb{Q}_p/\mathbb{Z}_p$ on this cover extends to an action of the much larger group of quasi-isogenies of

$$\widehat{\mathbb{G}}_m\times\mathbb{Q}_p/\mathbb{Z}_p.$$

This quasi-isogeny group contains a very large unipotent subgroup, the quasi-isogenies from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\widehat{\mathbb{G}_m}$, or, the universal cover $\widetilde{\widehat{\mathbb{G}_m}}$ in the language of Scholze-Weinstein [12]. This unipotent quasi-isogeny action is the ultimate source of the $\widehat{\mathbb{G}_m}$-action on M_{Katz}.

Remark 1.2.5. The action of a larger group of quasi-isogenies on this cover is a natural characteristic p analog of the prime-to-characteristic phenomenon where, when full level is added at $l\neq p$, there is an isogeny moduli interpretation that gives an action of $\mathrm{GL}_2(\mathbb{Q}_l)$ extending the action of $\mathrm{GL}_2(\mathbb{Z}_l)$ in the isomorphism moduli interpretation. Rigidifying in characteristic p using isomorphisms to an ordinary p-divisible group provides both more and less structure than when $l\neq p$: on the one hand, the isogeny group is solvable, and thus appears more like the subgroup of upper triangular matrices, but on the other hand the unipotent subgroup has a much richer structure than any groups that appear when $l\neq p$. If we instead rigidified using a height two formal group, we would obtain a super-singular Igusa variety, which has more in common with the $l\neq p$ case (the isogeny action is by the invertible elements of the non-split quaternion algebra over $\mathbb{Q}_p$); in [5] we use this structure to compare p-adic modular forms and continuous p-adic automorphic forms on the quaternion algebra ramified at p and ∞.

Remark 1.2.6. In this remark we explain a connection to perfectoid modular curves: The generic fiber of the big Igusa formal scheme is a twist of a component of the perfectoid ordinary locus over $\mathcal{O}_{\mathbb{C}_p}$. This component admits a natural action of the group of upper triangular matrices

$$\begin{pmatrix}1&\mathbb{Q}_p\\0&1\end{pmatrix}$$

which is identified over $\mathcal{O}_{\mathbb{C}_p}$ with an action of $\mathbb{Q}_p(1)$ on the generic fiber of the big Igusa formal scheme.

Using this, the action of the p-power roots of unity $\mathbb{Q}_p(1)/\mathbb{Z}_p(1)$, an infinite discrete set inside of the open ball $\widehat{\mathbb{G}_m}(\mathcal{O}_{\mathbb{C}_p})$, on functions on the generic fiber of $M_{\mathrm{Katz},\mathcal{O}_{\mathbb{C}_p}}$ can be identified with the action of the natural Hecke operators $\mathbb{Q}_p/\mathbb{Z}_p$ on the invariants under

$$\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

in functions on this component of the perfectoid ordinary locus. Thus, the $\widehat{\mathbb{G}_m}$ action extends the obvious action of $\mathbb{Q}_p/\mathbb{Z}_p$ to an action of a much larger group. We will not make any use of this connection to perfectoid modular curves in the present work.

1.2.7. *Local expansions.* An important aspect of our proof of Theorem A is that we make no appeal to local expansions at cusps or ordinary points, so that our approach is well-suited for generalization to other PEL Igusa varieties. After proving Theorem A, however, we also give a direction computation of the action on local expansions: we find that at ordinary points the action is given by multiplication of a Serre-Tate coordinate, and at the cusps it is given by multiplication of the inverse of the standard cuspidal coordinate q.

1.2.8. *Dwork's equation.* While developing some of the machinery used to compute the local expansions of the big Hecke action, and using the same philosophy of base change to a very ramified cover, we also give a new proof of Dwork's equation $\tau = \log q$ on the formal deformation space of $\widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p$ over $\overline{\mathbb{F}_p}$ which is valid for a larger family of *Kummer* p-divisible groups (which include not only the deformations of $\widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p$ over Artinian $\overline{\mathbb{F}_p}$-algebras, but also, e.g., the p-divisible group of the Tate curve, and other interesting groups when the base is not Artinian). These results can be found in Section 3.

1.2.9. *Other constructions.* After preparing an earlier version of this article, we learned that Gouvea had already some time ago [4, III.6.2] constructed a twisting measure equivalent to our $\widehat{\mathbb{G}_m}$-action (interpreted as an algebra action via p-adic Fourier theory as described in 1.2.10 below). In 5.6 we recall Gouvea's construction and explain how it can be rephrased as an alternate construction of the $\widehat{\mathbb{G}_m}$-action via the exotic isomorphisms of Katz [8, 5.6]. This construction is conceptually more opaque, but has the advantage of using only classical ideas.

There is a third, even simpler and even more opaque approach in which one builds the $\widehat{\mathbb{G}_m}$-action algebraically starting with the differential operator θ and the q-expansion principle; we explain this in Remark 1.2.11 below.

1.2.10. *The algebra action.* Via p-adic Fourier theory, the action of $\widehat{\mathbb{G}_m}$ described in Theorem A is equivalent to an action of $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$ on $\mathbb{V}_{\mathrm{Katz}}$. This action admits a particularly simple description on q-expansions: $f \in \mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$ acts as multiplication by $f(n)$ on the coefficient of q^n. As remarked above, the existence of this algebra action was first established by Gouvea [4, Corollary III.6.8], who interpreted it as a twisting measure.

From this perspective, the action of the monomial function z^k is by the derivation θ^k, and thus we may view our $\widehat{\mathbb{G}_m}$-action as interpolating the differential operators θ^k into an algebra action. In Section 6 we adopt this perspective to reinterpret some of the results of Katz [8] on two-variable Eisenstein measures.

Remark 1.2.11. In fact, we can *construct* the $\widehat{\mathbb{G}_m}$-action by applying the q-expansion principle [8, 5.2] to complete the action of polynomials in θ on $\mathbb{V}_{\mathrm{Katz}}$ to an action of $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$. Note that polynomials are not dense $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$, so the q-expansion principle needed here says not just that the q-expansion map is injective, but also that the cokernel is flat over $\mathbb{Z}_p$.

In order to use this method, one must first obtain θ without deducing its existence from the big Hecke action. One way to construct it is as the differential operator dual to the image of ω_{can}^2 under the Kodaira-Spencer isomorphism, which can be verified by a computation over $\mathbb{C}$, as explained by Katz [8, 5.8].

1.2.12. *Ordinary p-adic modular forms.* The action of $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$ interacts naturally with the $\mathbb{Z}_p^\times$ action on $\mathbb{V}_{\mathrm{Katz}}$, and thus we may view $\mathbb{V}_{\mathrm{Katz}}$ as a $\mathbb{Z}_p^\times$-equivariant quasi-coherent sheaf on the profinite set $\mathbb{Z}_p$ (viewed as a formal scheme whose ring of functions *is* $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$). As $\mathbb{Z}_p$ is the space of characters of $\widehat{\mathbb{G}_m}$, this viewpoint is analogous to thinking of functions on $Y_{\infty-\mathrm{ord}}$ in the global, archimedean setting as Fourier series.

In Section 7, we show that restriction induces an isomorphism between the fiber at $0 \in \mathbb{Z}_p$ of the subsheaf $\mathbb{V}_{\mathrm{Katz,hol}}$ of p-adic modular forms with holomorphic q-expansion and the space of ordinary p-adic modular forms à la Hida. Note that the fiber at zero is the maximal trivial quotient for the $\widehat{\mathbb{G}_m}$-action, and ordinary modular forms are those such that the corresponding p-adic Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ admits a map to a unitary principal series. Thus, our statement is a local, p-adic analog of the global, archimedean statement that the global automorphic representation representation attached to a classical modular form is globally induced if and only if its Fourier expansion has a non-zero constant term.

This interpretation of the space of ordinary modular forms is part of a larger connection between the space of functions on the big Igusa variety and (ϕ, Γ)-modules, which we will explore further in upcoming work.

1.2.13. *Generalizations.* There is a natural analogy in which functions on Igusa varieties are p-adic analogs of classical automorphic forms, and Theorem A provides a new tool for understanding and controlling these spaces using representation theory. We hope that the techniques introduced in this work will be useful in a more general setting, e.g., for studying differential operators on p-adic automorphic forms such as the μ-ordinary differential operators introduced by Eischen-Mantovan [3], or for studying functoriality in the p-adic Langlands program.

1.3. **Outline.** In Section 2 we collect some results on p-divisible groups that we will need in the rest of the paper. In Section 3 we study extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\widehat{\mathbb{G}_m}$; in particular, we introduce *Kummer p-divisible groups* and prove our generalization of Dwork's formula $\tau = \log q$.

In Section 4 we curate a zoo of moduli problems lying over the ordinary locus, and explain the relations between them. We apply these developments in Section 5 to construct the action of $\widehat{\mathbb{G}_m}$, prove Theorem A, and compute the action on local expansions.

Finally, in Section 6 we give an application to Eisenstein measures, and in Section 7 we give the application to ordinary p-adic modular forms.

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2. p-DIVISIBLE GROUPS

In this section we collect some results on p-divisible groups that will be useful in our construction. Our principal references are [11] and [12]; we also provide some complements.

For the proof Theorem A, the most important result in this section is Lemma 2.5.1, which, for I a nilpotent divided powers ideal in a ring R where p is nilpotent, computes the action of

$$\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p|_{R/I}, \widehat{\mathbb{G}_m}_{R/I})$$

on the Messing crystals evaluated on R.

2.1. **p-divisible groups.** Let R be a ring. A p-divisible group G over R is a sequence $G_1, G_2, \ldots$ of finite flat commutative group schemes G_i over R equipped with closed immersions $\iota_i : G_i \to G_{i+1}$ such that

(1) G_i is p^i-torsion
(2) ι_i identifies G_i with $G_{i+1}[p^n]$
(3) Multiplication by p is an fppf surjection $G_{i+1} \to G_{i+1}[p^i] = G_i$.

Each of the G_i defines a presheaf in abelian groups on $\mathrm{Alg}_R^{\mathrm{op}}$, and we will also denote by G the presheaf $\mathrm{colim} G_i$ so that

$$G(S) = \mathrm{colim} G_i(S) \tag{2.1.0.1}$$

for S an R-algebra. With this notation, we have a canonical identification $G_i = G[p^i]$.

Remark 2.1.1. Note that the maps are injective as maps of presheaves, so that in any faithful topology where the objects of $\mathrm{Alg}_R^{\mathrm{op}}$ are all quasi-compact (e.g. fppf), (2.1.0.1) is also the colimit as sheaves by [13, Lemma 7.17.5]. In particular, one could instead define a p-divisible group as, e.g., an fppf sheaf satisfying certain properties, as is often done in the literature. We prefer the given definition because we will have occasion later on to consider finer topologies.

Remark 2.1.2. We will usually consider p-divisible groups over a ring R where p is nilpotent, or over an affine formal scheme $\mathrm{Spf} R$ where p is topologically nilpotent in R. In the latter case, there are two natural ways one might try to define $G(S)$ for S a topological R-algebra: one could first algebraize to obtain a p-divisible group over $\mathrm{Spec} R$, then apply the definition above, or one could take the limit of $G(S/I)$ where I runs over the ideals defining the topology on R. The latter is the correct definition. For example, if $R = \mathcal{O}_{\mathbb{C}_p}$ with the p-adic topology and $G = \widehat{\mathbb{G}_m}$, then, the second, correct, definition gives $\widehat{\mathbb{G}_m}(R) = 1 + \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_{\mathbb{C}_p}$ while the first, incorrect, definition gives only the p-power roots of unity.

2.2. **Formal neighborhoods and Lie algebras.** For G a presheaf in abelian groups on $\mathrm{Alg}_R^{\mathrm{op}}$, we define the formal neighborhood of the identity $\widehat{G}$ by

$$\widehat{G}(S) = \ker G(S) \to G(S^{\mathrm{red}})$$

and the Lie algebra $\mathrm{Lie} G$ by

$$\mathrm{Lie} G(S) = \ker G(S[\epsilon]/\epsilon^2) \to G(S).$$

Note that, by definition $\mathrm{Lie}G(S) = \mathrm{Lie}\widehat{G}(S)$. We have the following important structural result:

Theorem 2.2.1. [11, Theorems 3.3.13 and 3.3.18] *If G is a p-divisible group over a ring R where p is nilpotent, then $\widehat{G}$ is a formal Lie group and G is formally smooth.*

2.3. **Universal covers.** For any presheaf in abelian groups G, we define

$$\widetilde{G} := \lim G \xleftarrow{p} G \xleftarrow{p} \ldots$$

and its sub-functor

$$T_pG := \lim 1 \xleftarrow{p} G[p] \xleftarrow{p} G[p^2] \ldots.$$

For $A \in \mathrm{Nilp}_R^{\mathrm{op}}$ we will write an element of $\widetilde{G}(A)$ as a sequence $(g_0, g_1, \ldots)$ such that $p(g_{i+1}) = g_i$ for all $i \geq 0$; the elements of T_pG are those such that $g_0 = 1$. In particular, we have an exact sequence of presheaves

$$1 \to T_pG \to \widetilde{G} \to G$$

where the map $\widetilde{G} \to G$ is $(g_0, g_1, \ldots) \mapsto g_0$.

When G is a p-divisible group, we call $\widetilde{G}$ the universal cover, following [12]. In this case, we have

Lemma 2.3.1. *If G is a p-divisible group,*

$$1 \to T_pG \to \widetilde{G} \to G \to 1 \tag{2.3.1.1}$$

is an exact sequence of sheaves in the fpqc topology.

Proof. We must verify that $\widetilde{G} \to G$ is surjective as a map of fpqc sheaves. Note that if $G_i = \mathrm{Spec}R_i$, then $\widetilde{G} \times_G G[p^n]$ is represented by $\mathrm{Spec}\,\mathrm{colim}_{i \geq n} R_i$, and the inclusion $R_n \to \mathrm{colim}R_i$ is an fpqc cover. Given an S-point $f : \mathrm{Spec}S \to \widetilde{G}$, which factors through $G[p^n]$ for some n, we find $\widetilde{S} = f^*\widetilde{G}$ is an fpqc cover of $\mathrm{Spec}S$ such that f is in the image of $\widetilde{G}(\widetilde{S})$. □

Remark 2.3.2. Exactness at the right in 2.3.1.1 typically fails in the fppf topology. For example, if $G = \widehat{\mathbb{G}_m}$ and R is finitely generated of characteristic p, then $\widetilde{\widehat{\mathbb{G}_m}}(R) = 1$. Any fppf cover of such an R is by finitely generated rings of characteristic p, thus $\widetilde{\widehat{\mathbb{G}_m}}$ is the trivial sheaf on the small fppf site of $\mathrm{Spec}R$. On the other hand, if R contains any nilpotents (e.g. $R = k[\epsilon]/\epsilon^2$), then $\widehat{\mathbb{G}_m}(R) \neq 1$, and thus the map $\widetilde{\widehat{\mathbb{G}_m}} \to \widehat{\mathbb{G}_m}$ is not surjective in the fppf topology.

2.3.3. *Crystalline nature of the universal cover.* Suppose G_0 is a p-divisible group over a ring R in which p is nilpotent, $R' \to R$ is a nilpotent thickening, and G is a lift of G_0 to R. Then, the reduction map

$$\widetilde{G}(R') \to \widetilde{G_0}(R)$$

is an isomorphism: the inverse sends $(g_0, g_1, \ldots)$ to $(g_0', g_1', \ldots)$ where g_i' is defined to be $p^n(\tilde{g}_{i+n})$ for n sufficiently large and any lift $\tilde{g}_{i+n} \in G(S)$ of g_{i+n}. Note that these lifts exist by the formal smoothness of Theorem 2.2.1 and the p^nth multiple is independent of lift for n sufficiently large by a lemma of Drinfeld [6, Lemma 1.1.2].

2.4. **The universal vector extension.** For R in which p is nilpotent, and G/R a p-divisible group, we denote by EG the universal vector extension of G, which is an extension

$$1 \to \omega_{G^\vee} \to EG \to G \to 1.$$

There is a natural map $s_G : \widetilde{G} \to EG$ sending $(g_0, g_1, \ldots) \in \widetilde{G}(S)$ to $p^n g_n'$ for n sufficiently large and g_n' any lift of g_n to $EG(S)$; this is well-defined since $\omega_{G^\vee}$ is annihilated by the same power of p that annihilates R.

Remark 2.4.1. From the construction of the universal vector extension in [11], we find that EG is the push-out of the extension 2.3.1.1 by the natural map $T_pG \to \omega_{G^\vee}$ sending x to $x^* \frac{dt}{t}$ where we think of x as a map from $G^\vee$ to $\widehat{\mathbb{G}_m}$. Note that the map $T_pG \to \omega_{G^\vee}$ factors through $G[p^n]$ for n sufficiently large (such that p^n annihilates R and thus $\omega_{G^\vee}$), so that EG can be constructed as an fppf pushout (avoiding issues with fpqc sheafification in showing the pushout exists). These considerations lead to the following question: is there a natural topology suitable for constructions such as in the previous remark involving T_pG and $\widetilde{G}$, but avoiding the set theoretic issues of the fpqc topology?

2.4.2. *Crystalline nature.* If $R \to R'$ is a nilpotent divided powers thickening, G_0 and H_0 are p-divisible groups over R, and G and H are lifts of G_0 and H_0, respectively, to R', and $\varphi : G_0 \to H_0$ is a morphism, then we obtain a morphism $E\varphi(R) : EG_0 \to EH_0$ by the universality of EG_0 (using that $\varphi^* EH_0$ is a vector extension of H_0). Messing [11, Theorem IV.2.2] shows that there is a functorial lift

$$EG \xrightarrow{E\varphi(R')} EH.$$

By [12, Lemma 3.2.2], the following diagram commutes:

(2.4.2.1)
$$\begin{tikzcd}
\tilde{G}(R') \arrow[r, "\sim"] \arrow[dd, "s_G"] & \tilde{G}_0(R) \arrow[r, "\tilde{\varphi}"] \arrow[d, "s_{G_0}"] & \tilde{H}_0(R) \arrow[r, "\sim"] \arrow[d, "s_{H_0}"] & \tilde{H}(R') \arrow[dd, "s_H"] \\
& EG_0(R) \arrow[r, "E\varphi(R)"] & EH_0(R) & \\
EG(R') \arrow[ur] \arrow[rrr, "E\varphi(R')"] & & & EH(R'). \arrow[ul]
\end{tikzcd}$$

In particular, if G_0/R is a p-divisible group, then, passing to Lie algebras, we obtain a (nilpotent) crystal in locally free $\mathcal{O}_{\text{crys}}$-modules DG_0 whose value on a nilpotent divided powers thickening $R \to R'$ is $\text{Lie}EG^\vee$ where G is any lift of G_0 to R'. Given such an R' and G, we obtain a filtered vector bundle on $\text{Spec}R'$

$$0 \to \omega_G \to \text{Lie}EG^\vee \to \text{Lie}G^\vee \to 0$$

with an integrable connection ∇_{crys}.

The assignment $G_0 \to DG_0$ is a contravariant functor: given $\varphi : G_0 \to H_0$ we obtain a map $DH_0 \to DG_0$ from the construction $E\varphi^\vee$.

2.5. **An important example.** We now explain how to compute the maps in diagram 2.4.2.1 when $G_0 = \mathbb{Q}_p/\mathbb{Z}_p$ and $H_0 = \widehat{\mathbb{G}_m}$.

For $G = \widetilde{\mathbb{Q}_p/\mathbb{Z}_p}$, $\widetilde{G} = \mathbb{Q}_p$. Then, $EG = \mathbb{Q}_p \times \mathbb{G}_a/\mathbb{Z}_p$, where we have identified $G^\vee$ with $\mathbb{G}_m$ and $\omega_{G^\vee}$ with $\mathbb{G}_a$ using the basis $\frac{dt}{t}$, and $\mathbb{Z}_p$ is included anti-diagonally, i.e. by $z \mapsto (z, -z)$. Here s_G is the map $a \mapsto (a, 0)$.

For $H = \mathbb{G}_m$, $EH = H$, and s_H is the map $\widetilde{\widehat{\mathbb{G}_m}} \to \widehat{\mathbb{G}_m}$ sending $(g_0, g_1, \ldots)$ to g_0.

A map from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\widehat{\mathbb{G}_m}$ over R is an element $(g_0, g_1, \ldots) \in T_p\widehat{\mathbb{G}_m}(R)$. Because $T_p\widehat{\mathbb{G}_m}(R) \subset \widetilde{\widehat{\mathbb{G}_m}}(R) \cong \widetilde{\widehat{\mathbb{G}_m}}(R')$ and the latter is a $\mathbb{Q}_p$-vector space, we obtain a unique map from $\mathbb{Q}_p$ to $\widetilde{\widehat{\mathbb{G}_m}}$. If we write $(g_0', g_1', \ldots)$ for the element of $\widetilde{\widehat{\mathbb{G}_m}}(R')$ lifting $(g_0, g_1, \ldots)$ (i.e. the image of $1 \in \mathbb{Q}_p$), then the map

$$E\mathbb{Q}_p/\mathbb{Z}_p(R') \xrightarrow{E\varphi(R')} E\widehat{\mathbb{G}_m}(R')$$

is induced by the map

$$\mathbb{Q}_p \times \mathbb{G}_a(R') \to \widehat{\mathbb{G}_m}, (z, x) \mapsto \tilde{\varphi}(z)_0 \cdot \exp(x \log(g_0')).$$

Here we have written $\tilde{\varphi}$ for the composition of the arrows at the top of the diagram (2.4.2.1) and the subscript 0 to denote its zeroth component, and the exponential and logarithm make sense because g_0' is congruent to 1 mod the kernel I of $R' \to R$, which is a nilpotent divided powers ideal. Because $\exp(z \log(g_0')) = (g_0')^z$ for $z \in \mathbb{Z}_p$, we find that the map is zero on the anti-diagonally embedded $\mathbb{Z}_p$. In particular, we deduce the following lemma, which we will use in our verification of Theorem A.

Lemma 2.5.1. *Suppose $(g_0, g_1, \ldots)$ is an element of $\widetilde{\widehat{\mathbb{G}_m}}(R)$ congruent to an element of $T_p\mathbb{G}_m(R/I)$ for a nilpotent divided powers ideal $I \subset R$. Then, the induced map*

$$\mathrm{Lie}E\mathbb{Q}_p/\mathbb{Z}_p = \mathbb{G}_a \cdot \frac{dt}{t} \to \mathbb{G}_a \cdot t\partial_t = \mathrm{Lie}E\mathbb{G}_m$$

is multiplication by $\log g_0$.

2.6. **Comparing the Gauss-Manin and crystalline connections.** Let S be a scheme where p is locally nilpotent, let $\pi : A \to S$ be an an abelian scheme, and write $A^\vee$ for the dual abelian scheme. We have the relative de Rham cohomology

$$V_{\mathrm{dR}} := R^1\pi_*\Omega^\bullet_{A/R}$$

with Hodge filtration

$$0 \to \omega_A \to V_{\mathrm{dR}} \to \mathrm{Lie}A^\vee \to 0.$$

We also have the universal extension of $EA[p^\infty]^\vee = EA^\vee[p^\infty]$

$$\omega_A \to EA[p^\infty]^\vee \to A[p^\infty]^\vee$$

and the induced Hodge filtration on $\mathrm{Lie}EA[p^\infty]^\vee$

$$0 \to \omega_A \to \mathrm{Lie}EA[p^\infty]^\vee \to \mathrm{Lie}A^\vee \to 0$$

(note we have identified ω_A with $\omega_{A[p^\infty]}$ and $\mathrm{Lie}A^\vee$ with $\mathrm{Lie}A^\vee[p^\infty] = \mathrm{Lie}A[p^\infty]^\vee$ via the natural maps).

Now, V_{dR} is equipped with the Gauss-Manin connection ∇_{GM}, and $\mathrm{Lie}A^\vee[p^\infty]$ is equipped with a connection ∇_{crys} via the crystalline nature of the universal vector extension. The work of Mazur-Messing [10] shows

Theorem 2.6.1. *There is a functorial isomorphism of filtered vector bundles with connection*

$$(\mathrm{Lie}EA[p^\infty]^\vee, \nabla_{\mathrm{crys}}) \cong (V_{\mathrm{dR}}, \nabla_{\mathrm{GM}}).$$

inducing the identity on the associated graded bundles for the Hodge filtrations.

Proof. The identity between $\mathrm{Lie}EA[p^\infty]^\vee$ with its Hodge filtration as constructed above and $\mathrm{LieExtrig}(A, \mathbb{G}_m)$ follows from the discussion of [10, I.2.6]. The stated isomorphism then follows from the results of [10, II.1]; in particular, the functoriality follows from [10, II.1.6]. □

2.6.2. *Connections and vector fields.* In preparation for our application of Theorem 2.6.1, we now recall the relation between some different perspectives on connections. We write $D = \mathrm{Spec}\mathbb{Z}[\epsilon]/\epsilon^2$, the dual numbers.

Given a vector bundle with connection (V, ∇) over S, and a vector field t, viewed as a map

$$t : D \times S \to S,$$

we obtain an isomorphism of vector bundles on $D \times S$

$$\nabla_t : t^* V_{\mathrm{dR}} \to 0^* V_{\mathrm{dR}}$$

where 0 is the zero vector field. It will be useful to make this isomorphism explicit when $S = \mathrm{Spec}R$ and M is the R-module of sections of V over $\mathrm{Spec}R$. Then the map t is given by

$$\begin{aligned} \alpha_t : R &\to R[\epsilon] \\ r &\mapsto r + dr(t)\epsilon \end{aligned}$$

and the zero section is given by

$$\begin{aligned} \alpha_0 : \quad R &\to R[\epsilon] \\ r &\mapsto r \end{aligned}$$

The isomorphism ∇_t is then given in coordinates by

$$\nabla_t : R[\epsilon] \otimes_{\alpha_t} M \to R[\epsilon] \otimes_{\alpha_0} M \tag{2.6.2.1}$$

$$1 \otimes m \mapsto 1 \otimes m + \epsilon \otimes \nabla_t(m). \tag{2.6.2.2}$$

where by abuse of notation we have also written ∇_t for the derivation $M \to M$ associated to t by ∇.

2.7. **Serre-Tate lifting theory.** For R a ring in which p is nilpotent, and $R_0 = R/I$ for I a nilpotent ideal, let

$$\mathrm{Def}(R, R_0)$$

be the category of triples

$$(E_0, G, \epsilon)$$

where E_0/R_0 is an elliptic curve, G is a p-divisible group, and $\epsilon : G|_{R_0} \xrightarrow{\sim} E_0[p^\infty]$ is an isomorphism.

We denote by $\mathrm{Ell}(R)$ the category of elliptic curves over R. There is a natural functor from $\mathrm{Ell}(R)$ to $\mathrm{Def}(R, R_0)$

$$E \mapsto (E_{R_0}, E[p^\infty], \epsilon_E) \tag{2.7.0.1}$$

where ϵ_E is the canonical isomorphism

$$E[p^\infty]_{R_0} \xrightarrow{\sim} E_{R_0}[p^\infty].$$

The following result is due to Serre-Tate, cf. [6, Theorem 1.2.1]:

Theorem 2.7.1. *The functor 2.7.0.1 is an equivalence of categories.*

3. Extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\widehat{\mathbb{G}_m}$

In this section we study extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\widehat{\mathbb{G}_m}$. In particular, we introduce Kummer p-divisible groups, and prove our generalization of Dwork's equation $\tau = \log q$ (Theorem 3.3.1 below).

3.1. **The canonical trivialization.** Suppose given an extension of p-divisible groups

$$\mathcal{E} : \widehat{\mathbb{G}_m} \to G \to \mathbb{Q}_p/\mathbb{Z}_p$$

over S with p locally nilpotent. The inclusion $\widehat{\mathbb{G}_m} \to G$ induces an isomorphism $\omega_G = \omega_{\widehat{\mathbb{G}_m}}$, and we denote by ω_{can} the image of $\frac{dt}{t}$ in ω_G. The map $G \to \mathbb{Q}_p/\mathbb{Z}_p$ induces an injection

$$\mathrm{Lie}E(\mathbb{Q}_p/\mathbb{Z}_p)^\vee = \mathrm{Lie}E\widehat{\mathbb{G}_m} = \mathrm{Lie}\widehat{\mathbb{G}_m} \to \mathrm{Lie}EG^\vee.$$

The image is the unit root filtration, which splits the Hodge filtration; we write u_{can} for the image of $t\partial_t \in \mathrm{Lie}\widehat{\mathbb{G}_m}$ in $\mathrm{Lie}EG^\vee$.

We thus obtain a trivialization

$$\mathrm{Lie}EG^\vee = \mathbb{G}_a \cdot \omega_{\mathrm{can}} \times \mathbb{G}_a \cdot u_{\mathrm{can}} \tag{3.1.0.1}$$

where the first term spans the Hodge filtration and the second the unit root filtration. The elements $t\partial_t$ and $\frac{dt}{t}$ are flat for the connections on $\mathrm{Lie}E(\mathbb{Q}_p/\mathbb{Z}_p)^\vee$ and $\mathrm{Lie}E(\widehat{\mathbb{G}_m})^\vee$, respectively, and thus we find that in the basis (3.1.0.1), ∇_{crys} is lower nilpotent, i.e.

$$\nabla_{\mathrm{crys}}(\omega_{\mathrm{can}}) \in u_{\mathrm{can}} \cdot \Omega_S,\ \nabla_{\mathrm{crys}}(t\partial_t) = 0.$$

In particular, the extension determines a differential form

$$d\tau_{\mathcal{E}} := \frac{\nabla_{\mathrm{crys}}(\frac{dt}{t})}{t\partial_t} \in \Omega_S.$$

The notation is a slight abuse, as in general there is no function $\tau_{\mathcal{E}}$ in $\mathcal{O}(S)$ whose differential is equal to $d\tau_{\mathcal{E}}$; nevertheless, as we will see below, it is natural to think of this as the differential of Dwork's divided powers coordinate τ.

3.2. **Kummer p-divisible groups.** For R a ring and $q \in R^\times$, we will construct an extension of p-divisible groups over $\mathrm{Spec}R$,

$$\mathcal{E}_q : \mu_{p^\infty} \to G_q \to \mathbb{Q}_p/\mathbb{Z}_p.$$

We call the extensions $\mathcal{E}_q$ arising from this construction *Kummer p-divisible groups.*

We first consider the fppf sheaf in groups

$$\mathrm{Roots}_q \subset \mathbb{G}_m \times \mathbb{Z}[1/p]$$

consisting of pairs (x, m) such that for k sufficiently large, $x^{p^k} = q^{p^k m}$.

Projection to the second component gives a natural map $\mathrm{Roots}_q \to \mathbb{Z}[1/p]$. The kernel is identified with μ_{p^∞}, and the projection admits a canonical section over $\mathbb{Z}$ by $1 \mapsto (q, 1)$. We consider the quotient by the image of this section

$$G_q := \mathrm{Roots}_q/\mathbb{Z}.$$

Lemma 3.2.1. *G_q is a p-divisible group, and the maps*

$$\mu_{p^\infty} \to \mathrm{Roots}_q \text{ and } \mathrm{Roots}_q \to \mathbb{Z}[1/p]$$

induce the structure of an extension

$$\mathcal{E}_q : \mu_{p^\infty} \to G_q[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p.$$

Proof. If we let Roots'_q be the subsheaf of Roots_q of elements (x, m) with $m \in \mathbb{Z}[1/p], 0 \leq m < 1$, then the group law induces an isomorphism

$$\mathrm{Roots}'_q \times \mathbb{Z} \to \mathrm{Roots}_q.$$

Thus, Roots'_q as a sheaf of sets is isomorphic to $\mathrm{Roots}_q/\mathbb{Z}$, and for A an R-algebra with $\mathrm{Spec}A$ connected,

$$G_q(A) = \mathrm{Roots}_q(A)/(q, 1)^{\mathbb{Z}}$$

and any element of $G_q(A)$ has a unique representative of the form $(x, m) \in \mathrm{Roots}_q(A)$ with $0 \leq m < 1$. Such an element is p^k-torsion if and only if $m \in 1/p^k\mathbb{Z}$ and $x^{p^k} = q^{p^k m}$. In particular, we find that $G_q = \mathrm{colim} G_q[p^k]$. Moreover, multiplication by p is an epimorphism because taking a pth root of x gives an fppf cover. Thus, to see that G_q is a p-divisible group, it remains only to see that $G_q[p]$ is a finite flat group scheme. In fact, for any k, our description of elements shows that $G_q[p^k]$ is represented by

$$\sqcup_{0 \leq a \leq p^k - 1} \mathrm{Spec} R[x]/(x^{p^k} - q^a),$$

with multiplication given by "carrying," i.e. for x_1 a root of q^{a_1} and x_2 a root of q^{a_2}, in the group structure

$$x_1 \cdot x_2 = \begin{cases} x_1 x_2 & \text{as a root of } q^{a_1+a_2} \text{ if } a_1 + a_2 < p^k \\ x_1 x_2/q & \text{as a root of } q^{a_1+a_2-p^k} \text{ if } a_1 + a_2 \geq p^k. \end{cases}$$

This is a finite flat group scheme.

Finally, the extension structure is clear from definition. □

Remark 3.2.2. Let $\mathrm{Roots}_{q,k} \subset \mathrm{Roots}_q$ be the elements (x, m) such that $p^k m \in \mathbb{Z}$ and $x^{p^k} = q^{p^k m}$, so that $\mathrm{Roots}_{q,k}/\mathbb{Z} = G_q[p^k]$. We have a natural pairing

$$\mathrm{Roots}_{q,k} \times \mathrm{Roots}_{q^{-1},k} \to \mu_{p^k}$$

given by $\langle (g, a), (h, b) \rangle = g^{p^k b} h^{p^k a}$, which induces a perfect pairing

$$G_q[p^k] \times G_{q^{-1}}[p^k] \to \mu_{p^k}.$$

It realizes an isomorphism of extensions

$$\mathcal{E}_q^\vee \xrightarrow{\sim} \mathcal{E}_{q^{-1}}$$

Note that at the level of groups $G_q \cong G_{q^{-1}}$; the extension structures $\mathcal{E}_q$ and $\mathcal{E}_{q^{-1}}$ differ by composition with an inverse on either $\mathbb{Q}_p/\mathbb{Z}_p$ or $\widehat{\mathbb{G}_m}$.

Example 3.2.3. The following three examples will be useful later on:

(1) For the Tate curve $\mathrm{Tate}(q)$ over $\mathbb{Z}((q))$, $\mathrm{Tate}(q)[p^\infty] = G_q[p^\infty]$. Indeed, our construction is modeled off of the p-divisible group of the Tate curve as realized via its rigid-analytic uniformization.

(2) For A an Artin local ring with perfect residue field k of characteristic p, any lift of the trivial extension $\widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p$ over k to A is uniquely isomorphic to $\mathcal{E}_q$ for a unique $q \in \widehat{\mathbb{G}}_m(A)$, and q^{-1} is the Serre-Tate coordinate of the lift (cf. Remark 3.2.6 below).
(3) The formation of $\mathcal{E}_q$ commutes with base change. In particular, there is a universal Kummer p-divisible group,
$$\mathcal{E}_{q_{\mathrm{univ}}}/\mathbb{G}_{m,\mathbb{Z}} = \mathrm{Spec}\mathbb{Z}[q_{\mathrm{univ}}^{\pm 1}],$$
so that for any $q \in R^\times$, $\mathcal{E}_q/\mathrm{Spec}R$ is given via pullback of $\mathcal{E}_{q_{\mathrm{univ}}}$ through the map $\mathrm{Spec}R \to \mathbb{G}_m$ given by $q \in R^\times = \mathbb{G}_m(R)$.

Remark 3.2.4. Over a general R, not every extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\widehat{\mathbb{G}_m}$ is a Kummer p-divisible group, and for those which are, there may not be a canonical choice of q as in the Artin local case. In particular, the extension given by the p-divisible group of the universal trivialized elliptic curve over $\mathbb{V}_{\mathrm{Katz},\mathbb{F}_p}$ is not a Kummer p-divisible group (cf. Remark 5.4.5 below).

Remark 3.2.5. For any $k \geq 0$, consider the Kummer sequence
$$\mu_{p^k} \to \mathbb{G}_m \xrightarrow{x \mapsto x^{p^k}} \mathbb{G}_m.$$
We may take the pull-back by
$$\mathbb{Z} \mapsto \mathbb{G}_m,\ 1 \mapsto q$$
to obtain an extension
$$\mu_{p^k} \to p^k - \mathrm{Roots}_q \to \mathbb{Z}.$$
Equivalently, this extension is the image of q under the coboundary map
$$\mathbb{G}_m(R) \to H^1_{\mathrm{fppf}}(\mathrm{Spec}R, \mu_{p^k}) = \mathrm{Ext}^1(\mathbb{Z}, \mu_{p^k}).$$
There is a natural map
$$p^k - \mathrm{Roots}_q \to \mathrm{Roots}_q.$$
Indeed, an element of $p^k - \mathrm{Roots}_q$ is a pair $(x,a) \in \mathbb{G}_m \times \mathbb{Z}$ such that $x^{p^k} = q^a$, and this is mapped to the pair
$$(x, a/p^k) \in \mathbb{G}_m \times \mathbb{Z}[1/p]$$
which lies in Roots_q. This is an isomorphism of $p^k - \mathrm{Roots}_q$ onto its image, which consists of all (x,m) such that $m \in \frac{1}{p^k}\mathbb{Z}$ and $x^{p^k} = q^{p^k m}$ – this is what we denoted by $\mathrm{Roots}_{q,k}$ in Remark 3.2.2. In particular, the map $\mathrm{Roots}_q \to G_q$ induces an isomorphism
$$p^k - \mathrm{Roots}_q/(q,p^k)^{\mathbb{Z}} \xrightarrow{\sim} G_q[p^k].$$
It is for this reason that we refer to $\mathcal{E}_q$ as a Kummer p-divisible group.

Note that there are also natural maps between the Kummer sequences as k varies inducing the obvious inclusions as sub-functors of Roots_q, and we find
$$\mathrm{Roots}_q = \mathrm{colim}_k p^k - \mathrm{Roots}_q.$$
To construct G_q we can also take the colimit already at the level of the Kummer sequences. If we do so, we obtain the (exact) exponential sequence
$$\mathcal{E}_{\exp}:\ \mu_{p^\infty} \to \mathbb{G}_m \to \mathrm{colim}\left(\mathbb{G}_m \xrightarrow{x \mapsto x^p} \mathbb{G}_m \xrightarrow{x \mapsto x^p} \dots\right).$$

There is a map

$$\alpha : \mathbb{Z} \to \mathbb{G}_m$$

sending 1 to q which extends uniquely to a map

$$\alpha_{1/p} : \mathbb{Z}[1/p] \to \operatorname{colim}\left(\mathbb{G}_m \xrightarrow{x\mapsto x^p} \mathbb{G}_m \xrightarrow{x\mapsto x^p} \dots\right).$$

Then, essentially by definition, $\alpha_{1/p}^*\mathcal{E}_{\exp}$ is the extension

$$\mu_{p^\infty} \to \mathrm{Roots}_q \to \mathbb{Z}[1/p].$$

The map $\alpha \times \mathrm{Id} : \mathbb{Z} \to \mathbb{G}_m \times \mathbb{Z}[1/p]$ factors through $\alpha_{1/p}^*\mathcal{E}_{\exp}$ and we find

$$\mathcal{E}_q = \alpha_{1/p}^*\mathcal{E}_{\exp}/\alpha \times \mathrm{Id}(\mathbb{Z}).$$

Remark 3.2.6. In this remark we explain a third construction of G_q and the connection to Serre-Tate coordinates: Consider the extension

$$\mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Q}_p/\mathbb{Z}_p. \tag{3.2.6.1}$$

We obtain an extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\mathbb{G}_m$, A_q, as the push-out of (3.2.6.1) by

$$\mathbb{Z} \to \mathbb{G}_m, \;\; 1 \mapsto q^{-1}. \tag{3.2.6.2}$$

We claim there is a natural isomorphism $G_q \cong A_q[p^\infty]$ respecting the extension structure. To see this, note that the push-out A_q is constructed as the quotient of $\mathbb{G}_m \times \mathbb{Z}[1/p]$ by the subgroup generated by $(q, 1)$. Then, the p^∞-torsion is just the image of Roots_q in A_q, as desired.

We note that if $q \in \widehat{\mathbb{G}_m}(R)$, then taking the push-out and passing to p^∞ torsion is equivalent to just taking the pushout under 3.2.6.2 viewed as a map to $\widehat{\mathbb{G}_m}$. Thus, when restricted to $q \in \widehat{\mathbb{G}_m}(R)$ for Artin local R with perfect residue field, our construction gives the extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by μ_{p^∞} with Serre-Tate coordinate q^{-1} (cf. [11, Appendix 2.4-2.5]).

We will need the following structural result on maps between Kummer p-divisible groups:

Lemma 3.2.7. *Isomorphisms $\mathcal{E}_q \xrightarrow{\sim} \mathcal{E}_{q'}$ are identified with the fiber above q'/q for the map*

$$\widetilde{\mathbb{G}_m} \to \mathbb{G}_m$$

sending $(x_0, x_1, \dots)$ to x_0.

Proof. Let $t = q'/q$. Suppose given a compatible system of roots t^{1/p^n} of t. We obtain an isomorphism between $G_q[p^n]$ and $G'_q[p^n]$ respecting the extension structure by sending an element $(a, k/p^n)$ to $(at^{k/p^n}, k/p^n)$, and these are compatible for varying n.

Conversely, given an isomorphism $\psi : G_q[p^\infty] \to G_{q'}[p^\infty]$ compatible with the extension structures, if we restrict to $\psi_n : G_q[p^n] \to G_{q'}[p^n]$, then for any $(a, 1/p^n) \in G_q[p^n]$, $\psi(a, 1/p^n) = (a', 1/p^n)$ for a' such that $a'^{p^n} = q'$, and a'/a is p^nth root of t that is *independent* of a because two choices of a differ by an element of μ_{p^n}; it thus comes from an element of $R^\times$, and the isomorphism at level p^n is as above; the roots of t chosen by varying the level then must also be compatible, giving an element of $\tilde{\mathbb{G}_m}$ mapping to t. $\square$

3.3. **Dwork's equation $\tau = \log q$.** The universal deformation of $\widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p$ over $\overline{\mathbb{F}_p}$, $G_{\mathrm{univ}}/\mathrm{Spf}W(\overline{\mathbb{F}}_p)[[t]]$, is canonically an extension

$$\mathcal{E} : \widehat{\mathbb{G}_m} \to G_{\mathrm{univ}} \to \mathbb{Q}_p/\mathbb{Z}_p.$$

Because $W(\overline{\mathbb{F}}_p)[[t]]$ is pro-Artin local,

$$\mathcal{E} = \mathcal{E}_q$$

for a unique $q \equiv 1 \mod (t,p)$, and q^{-1} is the Serre-Tate coordinate (cf. Example 3.2.3-(2) and Remark 3.2.6). The $W(\overline{\mathbb{F}_p})$ point x_{can} with $q = 1$ parameterizes the unique split lift to $W(\overline{\mathbb{F}}_p)$, the canonical lifting, and we can extend the canonical basis $\omega_{\mathrm{can}}|_{x_{\mathrm{can}}}, u_{\mathrm{can}}|_{x_{\mathrm{can}}}$ of $EG|_{x_{\mathrm{can}}}$ at this point to a flat basis over the divided powers envelope of x_{can} (the extension of $u_{\mathrm{can}}|_{x_{\mathrm{can}}}$ is just u_{can} itself, but ω_{can} is not flat so the flat extension of $\omega_{\mathrm{can}}|_{x_{\mathrm{can}}}$ is not equal to ω_{can}). The position of the Hodge filtration with respect to this basis then defines a divided powers function τ, and a conjecture of Dwork proven by Katz [6] states

$$\tau = \log q^{-1}.$$

As observed by Katz [7], this is equivalent to computing, in the language of 3.1,

$$d\tau_{\mathcal{E}_q} = d\log q^{-1}.$$

We now give a simple proof of this result by using a very ramified base-change to split $\mathcal{E}_q$. The result is valid for any Kummer p-divisible group:

Theorem 3.3.1. *For S a scheme on which p is locally nilpotent and $q \in \mathbb{G}_m(S) = \mathcal{O}(S)^\times$,*

$$d\tau_{\mathcal{E}_q} = -d\log q = d\log q^{-1} = -\frac{dq}{q}$$

Proof. By reduction to the universal case, it suffices to prove this for $\mathcal{E}_q$ over

$$S = \mathbb{G}_{m,\mathbb{Z}/p^n\mathbb{Z}} = \mathrm{Spec}\mathbb{Z}/p^n\mathbb{Z}[q^{\pm 1}].$$

In this case, Ω_S is free with basis $d\log q = \frac{dq}{q}$, thus it suffices to show that

$$\nabla_{\mathrm{crys}, q\partial_q}(d\tau_{\mathcal{E}_q}) = -1.$$

The vector field $q\partial q$, thought of as a map

$$t : D \times S \to S$$

is given by the map of rings

$$R \to R[\epsilon]/\epsilon^2,\ q \mapsto (1+\epsilon)q,$$

and we can compute the isomorphism

$$t^*\mathrm{Lie}EG_q^\vee \to 0^*\mathrm{Lie}EG_q^\vee$$

induced by ∇_{crys} as follows:

First, we observe that $t^*\mathcal{E}_q = \mathcal{E}_{(1+\epsilon)q}$ and $0^*\mathcal{E}_q = \mathcal{E}_q$, where q is thought of an element of $R[\epsilon]$, and under these identifications the isomorphism

$$0^*\mathcal{E}_q \mod \epsilon \xrightarrow{\sim} t^*\mathcal{E}_q \mod \epsilon$$

is identified with the canonical isomorphism

$$\mathcal{E}_q \mod \epsilon = \mathcal{E}_{(1+\epsilon)q} \mod \epsilon \tag{3.3.1.1}$$

given by $(1+\epsilon)q = q \mod \epsilon$.

Thus, using the description of 2.6.2, it suffices to show that the induced map

$$\nabla_{\mathrm{crys},q\partial_q} : \mathrm{Lie} EG^\vee_{(1+\epsilon)q} \to \mathrm{Lie} EG^\vee_q \tag{3.3.1.2}$$

is given in the canonical bases by

$$\begin{pmatrix} 1 & 0 \\ 1-\epsilon & 1 \end{pmatrix}. \tag{3.3.1.3}$$

It suffices to verify this after flat base change, so we may adjoin roots q^{1/p^∞} and $(1+\epsilon)^{1/p^\infty}$ to obtain a ring $R_\infty/(R[\epsilon]/\epsilon^2)$.

Over R_∞, the maps $1/p^n \to q^{1/p^n}$ and $1/p^n \to (1+\epsilon)^{1/p^n} q^{1/p^n}$ split $\mathcal{E}_q$ and $\mathcal{E}_{(1+\epsilon)q}$. In these trivializations, the canonical isomorphism (3.3.1.1) is identified with the map

$$\widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p \to \widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p$$

given by

$$\begin{pmatrix} 1 & ((1+\epsilon)^{-1}, (1+\epsilon)^{-1/p}, \ldots)) \\ 0 & 1 \end{pmatrix} \mod \epsilon$$

The transpose map

$$G^\vee_{(1+\epsilon)q} \to G^\vee_q$$

is identified with

$$\begin{pmatrix} 1 & 0 \\ ((1+\epsilon)^{-1}, (1+\epsilon)^{-1/p}, \ldots) & 1 \end{pmatrix} \mod \epsilon,$$

and using Theorem 5.4.1, we concluded that over R_∞, in the canonical bases the map (3.3.1.2) is given by (3.3.1.3), as desired. □

4. Moduli problems for ordinary elliptic curves

In this section, we discuss various moduli problems for ordinary elliptic curves over a base S where p is locally nilpotent.

4.1. **The big Igusa, Igusa, and Katz moduli problems.**

4.1.1. *Adelic Tate modules.* For T a topological space, we write $\underline{T}$ for the functor on Sch sending S to $\mathrm{Cont}(|S|, T)$.

Given an elliptic curve E/S over a scheme S, we define the prime-to-p Tate module

$$T_{\widehat{\mathbb{Z}}^{(p)}} E := \lim_{(n,p)=1} E[n],$$

as a functor on Sch$/S$, where the transition map from $E[n']$ to $E[n]$ for $n|n'$ is multiplication by n'/n. The transition maps are affine, so the prime-to-p Tate module is representable. We define the adelic prime-to-p Tate module as the sheaf on S_{Zar}

$$V_{\mathbb{A}_f^{(p)}} E := T_{\widehat{\mathbb{Z}}^{(p)}} E \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The prime-to-p Tate module is functorial for quasi-p-isogenies, and the prime-to-p adelic Tate module is functorial for quasi-isogenies.

4.1.2. *Structures at p when p is nilpotent.* If R is a ring in which p is nilpotent, and $E/\mathrm{Spec}R$ is an elliptic curve, we will consider the following functors on $\mathrm{Nilp}_R^{\mathrm{op}}$:

(1) The formal group $\widehat{E}$ (as defined already in 2.2),
(2) The p-divisible group
$$E[p^\infty] := \mathrm{colim} E[p^n]$$
(3) The universal cover of $E[p^\infty]$,
$$\widetilde{E[p^\infty]} = \lim_p E[p^\infty]$$
(as defined already in 2.3.)

The formal group and p-divisible group of E are functorial with respect to quasi-prime-to-p-isogenies of E, and the universal cover of $E[p^\infty]$ is functorial with respect to quasi-isogenies of E.

4.1.3. *The big Igusa moduli problem.* The big Igusa moduli problem $M_{\mathrm{big\,Igusa}}$, classifies, for $\mathrm{Spec}R \in \mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ the set of triples
$$(E, \widetilde{\varphi}, \alpha)$$
where E/R is an elliptic curve,
$$\widetilde{\varphi} : \widetilde{E[p^\infty]} \xrightarrow{\sim} \widetilde{\mu_{p^\infty}} \times \mathbb{Q}_p,$$
and α is an isomorphism of Zariski sheaves on $\mathrm{Spec}R$
$$\alpha : \underline{(\mathbb{A}_f^{(p)})^2} \xrightarrow{\sim} V_{\mathbb{A}_f^{(p)}} E,$$
all considered up to quasi-isogeny of E.

The data we consider is rigid, that is, any two triples $(E, \widetilde{\varphi}, \alpha)$ and $(E', \widetilde{\varphi}', \alpha')$ representing x differ by a *unique* quasi-isogeny. Thus we obtain an elliptic curve up-to-isogeny E_x on $\mathrm{Spec}R$. Because quasi-isogenies induce isomorphisms on $\widetilde{E[p^\infty]}$, we deduce that given $x \in M_{\mathrm{big\,Igusa}}(R)$, the universal cover $\widetilde{E_x[p^\infty]}$ is well-defined.

4.1.4. *The Katz and Igusa moduli problems.* The Katz (resp. Igusa) moduli problem M_{Katz} (resp. M_{Igusa}), classifies, for $\mathrm{Spec}R \in \mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ the set of triples (resp. quadruples)
$$(E, \widehat{\varphi}, \alpha) \text{ (resp. } (E, \widehat{\varphi}, \varphi^{\text{ét}}, \alpha))$$
where
$$\widehat{\varphi} : \widehat{E} \xrightarrow{\sim} \mu_{p^\infty},$$
(resp. and
$$\varphi^{\text{ét}} : E[p^\infty]/\widehat{E} \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p),$$
and α is an isomorphism of Zariski sheaves on $\mathrm{Spec}R$
$$\alpha : \underline{(\mathbb{A}_f^{(p)})^2} \xrightarrow{\sim} V_{\mathbb{A}_f^{(p)}} E,$$
all considered up to quasi-prime-to-p-isogeny of E.

Again, in both cases the data we consider is rigid, so any tuples representing a point $x \in M_{\mathrm{Katz}}(R)$ or (resp. $x \in M_{\mathrm{Igusa}}(R)$) differ by a unique quasi-prime-to-p isogeny, and thus we obtain an elliptic curve up-to-prime-to-p-isogeny E_x over $\mathrm{Spec}R$. Because quasi-prime-to-p isogenies induce isomorphisms on p-divisible groups, we obtain a well-defined p-divisible group $E_x[p^\infty]$ on $\mathrm{Spec}R$.

We note that to give the data $\widehat{\varphi}$ and $\varphi^{\text{ét}}$ is equivalent to equipping $E[p^\infty]$ with the structure of an extension

$$\mathcal{E}_{E[p^\infty],\widehat{\varphi},\varphi^{\text{ét}}} : \widehat{\mathbb{G}_m} \to E[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p. \tag{4.1.4.1}$$

In particular, given $x \in M_{\text{Igusa}}(R)$, we obtain a well-defined extension

$$\mathcal{E}_x : \widehat{\mathbb{G}_m} \to E_x[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p. \tag{4.1.4.2}$$

4.1.5. *Distinguishing representatives.* Standard arguments show that any $x \in M_{\text{big Igusa}}(R)$ can be represented by a triple $(E, \widetilde{\varphi}, \alpha)$ such that $\widetilde{\varphi}$ comes from an isomorphism

$$\varphi : E[p^\infty] \xrightarrow{\sim} \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p.$$

We call such a triple a *distinguished* representative for x. For any two distinguished representatives $(E, \widetilde{\varphi}, \alpha)$ and $(E', \widetilde{\varphi}', \alpha')$, there is a unique quasi-prime-to-p-isogeny

$$f : E \to E'$$

relating the two triples, i.e. such that

$$\widetilde{\varphi} = \widetilde{\varphi}' \circ f_* \text{ and } \alpha = (f^{-1})_* \circ \alpha'.$$

Similarly, if we fix a $\widehat{\mathbb{Z}}^{(p)}$-lattice $\mathcal{L}$ in $(\mathbb{A}_f^{(p)})^2$, any $x \in M_{\text{big Igusa}}(R)$ can be represented by a distinguished triple $(E, \widetilde{\varphi}, \alpha)$ such that α restricts to an isomorphism

$$\underline{\mathcal{L}} \xrightarrow{\sim} T_{\widehat{\mathbb{Z}}^{(p)}} E.$$

We call such a triple an $\mathcal{L}$-*distinguished* representative for x. For any two $\mathcal{L}$-distinguished representatives $(E, \widetilde{\varphi}, \alpha)$ and $(E', \widetilde{\varphi}', \alpha')$, there is a unique isomorphism

$$f : E \xrightarrow{\sim} E'$$

relating the two triples.

Similarly, if $x \in M_{\text{Katz}}(R)$ or $x \in M_{\text{Igusa}}(R)$ then x can be represented by a tuple $(E, \ldots, \alpha)$ such that α restricts to an isomorphism

$$\underline{\mathcal{L}} \xrightarrow{\sim} T_{\widehat{\mathbb{Z}}^{(p)}} E,$$

and we call such a tuple an $\mathcal{L}$-distinguished representative for x, and any two $\mathcal{L}$-distinguished representatives are related by an isomorphism of the curve.

4.2. **Group actions.**

4.2.1. *Automorphism groups at p.* Let

$$\begin{aligned} B_p &:= \operatorname{Aut}\left(\widetilde{\mu_{p^\infty}} \times \mathbb{Q}_p\right), \\ M_p &:= \operatorname{Aut}(\widetilde{\mu_{p^\infty}}) \times \operatorname{Aut}(\mathbb{Q}_p) = \underline{\mathbb{Q}_p^\times} \times \underline{\mathbb{Q}_p^\times} \\ N_p &:= \operatorname{Hom}(\mathbb{Q}_p, \widetilde{\mu_{p^\infty}}) = \widetilde{\mu_{p^\infty}} \end{aligned}$$

There is a natural projection $B_p \to M_p$. There are also natural inclusions

$$M_p \hookrightarrow B_p,\ N_p \hookrightarrow B_p,$$

and $B_p = N_p \rtimes M_p$

We also write

$$\begin{aligned} B_p^\circ &:= \operatorname{Aut}\left(\mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p\right), \\ M_p^\circ &:= \operatorname{Aut}(\mu_{p^\infty}) \times \operatorname{Aut}(\mathbb{Q}_p/\mathbb{Z}_p) = \underline{\mathbb{Z}_p^\times} \times \underline{\mathbb{Z}_p^\times} \\ N_p^\circ &:= \operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) = T_p\mu_{p^\infty}. \end{aligned}$$

Again there are natural inclusions

$$M_p^\circ \hookrightarrow B_p^\circ,\ N_p^\circ \hookrightarrow B_p^\circ$$

and

$$B_p^\circ = N_p^\circ \rtimes M_p^\circ.$$

The natural isomorphism

$$\widetilde{\mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p} = \widetilde{\mu_{p^\infty}} \times \widetilde{\mathbb{Q}_p/\mathbb{Z}_p} = \widetilde{\mu_{p^\infty}} \times \mathbb{Q}_p$$

induces inclusions

$$M_p^\circ \hookrightarrow M_p,\ B_p^\circ \hookrightarrow B_p,\ N_p^\circ \hookrightarrow N_p,$$

and these are compatible with the various inclusions above.

4.2.2. *Action on the moduli problems.* We write $G^p = \underline{\mathrm{GL}_2(\mathbb{A}_f^{(p)})}$. We have natural actions by composition with the level structures of

- $B_p \times G^p$ on $M_{\text{big Igusa}}$,
- $M_p^\circ \times G^p$ on M_{Igusa},
- and $\underline{\mathbb{Z}_p^\times} \times G^p$ on M_{Katz}

where for the action on M_{Katz} we make the natural identification

$$\underline{\mathbb{Z}_p^\times} = \mathrm{Aut}(\mu_{p^\infty}).$$

4.2.3. *Action of the unipotent subgroup on distinguished representatives.* For

$$n \in N_p(R) = \widetilde{\mu_{p^\infty}}(R) \text{ and } x \in M_{\text{big Igusa}}(R),$$

it will be useful when we do computations with the crystalline connection for us to have a more explicit description of $n \cdot x$ on the distinguished representatives of 4.1.5. We give such a description now; the key point is that any unipotent automorphism of the universal cover lifts a unipotent automorphism of the p-divisible group $\widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p$ modulo a nilpotent ideal.

Write $n = (\zeta_k) \in \widetilde{\mu_{p^\infty}}(R)$ and let I be any nilpotent ideal of R containing $\zeta_0 - 1$. Then,

$$n \mod I = (1, \zeta_1 \mod I, \zeta_2 \mod I, \ldots)$$

is an element of $N_p^\circ(R/I)$. Now, if $(E, \widetilde{\varphi}, \alpha) \in M_{\text{big Igusa}}(R)$ is a distinguished representative for x such that $\widetilde{\varphi}$ comes from an isomorphism

$$\varphi : E[p^\infty] \xrightarrow{\sim} \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p,$$

then

$$n \cdot (E, \varphi, \alpha) = \left(E', \widetilde{\varphi'}, \alpha'\right)$$

where E' is the Serre-Tate lift from R/I to R of $E_{R/I}$ determined by the isomorphism

$$(n \mod I) \circ \varphi_{R/I} : E_{R/I}[p^\infty] \xrightarrow{\sim} (\mathbb{Q}_p/\mathbb{Z}_p \times \widehat{\mathbb{G}_m})_{R/I},$$

φ' is the natural isomorphism $E'[p^\infty] \xrightarrow{\sim} (\mathbb{Q}_p/\mathbb{Z}_p \times \widehat{\mathbb{G}_m})$, and α' is the unique lift of $\alpha|_{R/I}$ from $E'_{R/I}$ to E'.

4.2.4. *Some characters.* We write $\det : B_p \to \underline{\mathbb{Q}_p^\times}$ for the character given by projection to M_p composed with multiplication of the two factors; we will also write det for the determinant on $\mathrm{GL}_2(\mathbb{Q}_l)$ for $l \neq p$. As in the introduction, we write

$$\begin{aligned} \det{}_{\mathrm{ur}} : G^p &\to \underline{\mathbb{Z}[1/p]^\times} \\ (g_l)_{l\neq p} &\mapsto \prod_{l \neq p} |\det(g_l)|_l \end{aligned}$$

We will also need the character

$$\begin{aligned} \det{}_p : B_p \times G^p &\to \underline{\mathbb{Z}_p^\times} \\ (g_p, g^p) = (g_l)_l &\mapsto \det(g_p) \prod_l |\det g_l| \\ &= \det(g_p)|\det(g_p)| \prod_{l\neq p} |\det g_l| \\ &= \det(g_p)|\det(g_p)|\det{}_{\mathrm{ur}}(g^p). \end{aligned}$$

4.3. **Projection from big Igusa to Igusa level.** There is a natural projection map

$$M_{\text{big Igusa}} \to M_{\text{Igusa}} \tag{4.3.0.1}$$

defined as follows: given

$$x \in M_{\text{big Igusa}}(R),$$

take a distinguished representative $(E, \widetilde{\varphi}, \alpha)$ for x as in 4.1.5 and write φ for the isomorphism inducing $\widetilde{\varphi}$. We obtain a point of $M_{\text{Igusa}}(R)$ determined by E, α, and the graded parts of φ, which is well defined because any two distinguished representatives differ by a quasi-prime-to-p-isogeny. We define the projection of x to be this point.

4.3.1. *Interaction with group action.* The projection map (4.3.0.1) is equivariant for the group actions of $B_p^\circ \times G^p \subset B_p \times G^p$ on $M_{\text{big Igusa}}$ and of $M_p^\circ \times G^p$ on M_{Igusa} under the natural projection from B_p° to M_p° (with kernel N_p°).

4.3.2. *Preimages under projection.* If $y \in M_{\text{Igusa}}(R)$, then from our construction we find that the choice of a preimage $x \in M_{\text{big Igusa}}(R)$ under the projection (4.3.0.1) is equivalent to the choice of a splitting of the extension $\mathcal{E}_y$ (defined in (4.1.4.2)). Two such splittings differ by a unique unipotent automorphism of $\mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$, and thus two preimages x and x' of y differ by the action of a unique element of $N_p^\circ(R)$. In fact,

Lemma 4.3.3. *The projection map 4.3.0.1) realizes $M_{\text{big Igusa}}$ as an fpqc N_p°-torsor*[1] *over M_{Igusa}.*

[1]We will explain later that these moduli problem are representable, but at the current point in the exposition we have not yet shown this, so we do not know yet that these are fpqc sheaves. Thus, in the mean time this should be interpreted as saying two elements with the same image in $M_{\text{Igusa}}(R)$ differ by a unique element of N_p° and every element of $M_{\text{Igusa}}(R)$ admits a pre-image after restriction to an fpqc-cover of $\mathrm{Spec} R$.

Proof. It remains only to see that the map is surjective in the fpqc topology. If

$$(E, \widehat{\varphi}, \varphi^{\text{ét}}, \alpha) \in M_{\text{Igusa}}(R)$$

then we construct a cover $\text{Spec}R_\infty$ of $\text{Spec}R$ as a limit of the covers $\text{Spec}R_n$ parameterizing splittings of the extension

$$\mathcal{E}_{E[p^\infty], \widehat{\varphi}, \varphi^{\text{ét}}}[p^n].$$

These are finite flat over R – indeed, the cover at level n is the fiber of

$$E[p^n] \to 1/p^n\mathbb{Z}/\mathbb{Z}$$

over $1/p^n + \mathbb{Z}$. Thus, the limit $\text{Spec}R_\infty \to \text{Spec}R$ is an fpqc cover, and there is a canonical splitting of $\mathcal{E}_{E[p^\infty], \widehat{\varphi}, \varphi^{\text{ét}}}$ over $\text{Spec}R_\infty$, giving the pre-image. □

4.4. **The Weil pairing and components.**

4.4.1. *The Weil pairing and the universal cover.* For R a ring in which p is nilpotent and $E/\text{Spec}R$ an elliptic curve, the p^n-Weil pairing is a perfect antisymmetric pairing

$$e_{p^n,E} : E[p^n] \times E[p^n] \to \mu_{p^n}.$$

It induces an anti-symmetric $\mathbb{Q}_p$-bilinear pairing

$$\widetilde{e}_E : \widetilde{E[p^\infty]} \times \widetilde{E[p^\infty]} \to \widetilde{\mu_{p^\infty}}$$

given by

$$\widetilde{e}_E((a_k), (b_k)) = (c_k)$$

where

$$c_k = (e_{p^t,E}(a_i, b_j))^{p^s}$$

for $i + j = s + t + k$ and t large enough that $a_i, b_j \in E[p^t]$ so that the right-hand side is defined.

Lemma 4.4.2. *If $f : E \to E'$ is an isogeny or quasi-prime-to-p isogeny, then*

$$f^* e_{p^n,E'} = e_{p^n,E}^{\deg f}.$$

If f is a quasi-isogeny,

$$f^* \widetilde{e}_{E'} = \widetilde{e}_E^{\deg f}.$$

Proof. The first equation for isogenies is a well-known property of the Weil pairing, and the second equation for isogenies is then immediate from the definition of $\widetilde{e}$. Once the isogeny statements are established, the quasi-isogeny statements follow as raising to a prime-to-p integer power is invertible on μ_{p^n} and raising to any integer power is invertible on $\widetilde{\mu_{p^\infty}}$. □

In particular, we note that the p^n Weil pairings e_{p^n} are functorial in degree one quasi-prime-to-p-isogenies of E, and the universal cover Weil pairing $\widetilde{e}$ is functorial in degree one quasi-isogenies of E.

4.4.3. *Fixing a Weil-Pairing.* Given $x \in M_{\text{big Igusa}}(R)$, there is no canonical choice of a Weil pairing on $\widetilde{E_x[p^\infty]}$ because the Weil pairings are not preserved by quasi-isogenies. However, if we fix a $\widehat{\mathbb{Z}}^{(p)}$-lattice $\mathcal{L}$ in $(\mathbb{A}_f^{(p)})^2$ and consider an $\mathcal{L}$-distinguished representative (E, φ, α) for x as described in 4.1.5, we obtain a well-defined Weil pairing on $\widetilde{E_x[p^\infty]}$. Moreover, this Weil pairing only depends on the equivalence class $[\mathcal{L}]$ of $\mathcal{L}$, where two lattices are equivalent if they both contain a third lattice with the same index. We denote this Weil-pairing on $\widetilde{E_x}[p^\infty]$ by $\widetilde{e}_{x,[\mathcal{L}]}$.

We may proceed similarly from a choice of equivalence classes of lattices $[\mathcal{L}]$ to obtain Weil pairings $e_{p^n,x,[\mathcal{L}]}$ on $E_x[p^n]$ for $x \in M_{\text{Igusa}}(R)$ or $x \in M_{\text{Katz}}(R)$.

4.4.4. *The p-component map.* Given a point $x \in M_{\text{big Igusa}}(R)$, we obtain as above a Weil pairing $\widetilde{e}_{x,[\mathcal{L}]}$ on $\widetilde{E_x[p^\infty]}$, and thus, via pull-back through $\widetilde{\varphi}$, a non-degenerate alternating pairing

$$\widetilde{\varphi}^* : \widetilde{e}_{x,[\mathcal{L}]}(\widetilde{\mu_{p^\infty}} \times \mathbb{Q}_p)^2 \to \widetilde{\mu_{p^\infty}}.$$

Any such pairing is a $\underline{\mathbb{Q}_p^\times}(R)$ multiple of the standard pairing

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1^{y_2} y_1^{-x_2}.$$

In our case it is even a $\underline{\mathbb{Z}_p^\times}(R)$-multiple because to obtain our Weil pairing we chose a representing triple where $\widetilde{\varphi}$ came from an isomorphism $E[p^\infty] \xrightarrow{\sim} \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$. Thus we obtain a map

$$c_{\text{big Igusa},[\mathcal{L}]} : M_{\text{big Igusa}} \to \underline{\mathbb{Z}_p^\times}$$

by remembering only this scaling factor; we call this the p-component map (with respect to $[\mathcal{L}]$).

Similarly, given a point $x \in M_{\text{Igusa}}(R)$, we obtain as above Weil pairings $e_{p^n,x,[\mathcal{L}]}$ on $E_x[p^n]$. We obtain an automorphism of μ_{p^∞} given on μ_{p^n} by

$$\zeta \mapsto e_{p^n,x,[\mathcal{L}]}\left(\widehat{\varphi}(\zeta), \varphi^{\text{ét}}(1/p^n)\right),$$

and this automorphism is given by a unique element of $\underline{\mathbb{Z}_p^\times}(R)$. This gives a map

$$c_{\text{Igusa},[\mathcal{L}]} : M_{\text{Igusa}} \to \underline{\mathbb{Z}_p^\times},$$

which we also call the p-component map. It is straightforward to check that the following diagram commutes (where the vertical map is given by the projection (4.3.0.1)):

(4.4.4.1)
$$\begin{tikzcd}
M_{\text{big Igusa}} \arrow[d] \arrow[dr, "c_{\text{big Igusa},[\mathcal{L}]}"] & \\
M_{\text{Igusa}} \arrow[r, "c_{\text{Igusa},[\mathcal{L}]}"'] & \underline{\mathbb{Z}_p^\times}
\end{tikzcd}$$

Remark 4.4.5. We can view $\underline{\mathbb{Z}_p^\times}$ as the formal scheme incarnation of the profinite set $\mathbb{Z}_p^\times$; it is represented by $\mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$. Over $\overline{\mathbb{F}_p}$, if we choose a lattice $\mathcal{L}$, the map $c_{\text{big Igusa},[\mathcal{L}]}$ combined with the Weil pairings away from p gives a natural identification of the connected components of $M_{\text{big Igusa},\overline{\mathbb{F}_p}}$ (and $M_{\text{Igusa},\overline{\mathbb{F}_p}}$) with $\mathbb{Z}_p^\times \times \widehat{\mathbb{Z}^{(p)}}^\times(1)$. In particular, over $\mathbb{F}_p$, the set $\mathbb{Z}_p^\times$ is close to being the set of connected components of $M_{\text{big Igusa},\mathbb{F}_p}$; there is a finite discrepancy coming from roots of unity in $\mathbb{F}_p$. The same is true for M_{Igusa}.

4.4.6. *Equivariance of the p-component map.*

Lemma 4.4.7. *The map*

$$c_{\text{big Igusa},[\mathcal{L}]} : M_{\text{big Igusa}} \to \underline{\mathbb{Z}_p^\times}$$

is $B_p \times G^p$-equivariant when $M_{\text{big Igusa}}$ is equipped with the natural action and $\underline{\mathbb{Z}_p^\times}$ is equipped with the (left) action of multiplication by $\det_p^{-1}$.

Similarly, the map

$$c_{\text{Igusa},[\mathcal{L}]} : M_{\text{Igusa}} \to \underline{\mathbb{Z}_p^\times}$$

is $M_p^\circ \times G^p$-equivariant when M_{Igusa} is equipped with the natural action and $\underline{\mathbb{Z}_p^\times}$ is equipped with the (left) action of multiplication by $\det_p^{-1}$.

Proof. We first treat the case of $M_{\text{big Igusa}}$. For $g = g_p \times g^{(p)} \in B_p \times G^p(R)$ and $x \in M_{\text{big Igusa}}(R)$, we represent x by a $\mathcal{L}$-distinguished triple $(E, \widetilde{\varphi}, \alpha)$ as in 4.1.5. Then $g^{-1} \cdot x$ is represented by a $\mathcal{L}$-distinguished triple $(E', \widetilde{\varphi}', \alpha')$ where there is a quasi-isogeny $f : E \to E'$ such that the following two diagrams commute:

$$\begin{tikzcd}
(\mathbb{A}_f^{(p)})^2 \arrow[r, "g^{(p)}"] \arrow[d, "\alpha'"] & (\mathbb{A}_f^{(p)})^2 \arrow[d, "\alpha"] & \widetilde{\widehat{\mathbb{G}}_m} \times \mathbb{Q}_p \arrow[d, "\widetilde{\varphi}^{-1}"] & \widetilde{\widehat{\mathbb{G}}_m} \times \mathbb{Q}_p \arrow[l, "g_p"'] \arrow[d, "(\widetilde{\varphi}')^{-1}"] \\
V_{\mathbb{A}_f^{(p)}} E' & V_{\mathbb{A}_f^{(p)}} E \arrow[l, "f"'] & \widetilde{E[p^\infty]} \arrow[r, "f"] & \widetilde{E'[p^\infty]}
\end{tikzcd} \tag{4.4.7.1}$$

We deduce from commutativity of the diagrams (4.4.7.1) that

$$\deg f = |\det(g_p)|_p \det_{\text{ur}}(g^{(p)}). \tag{4.4.7.2}$$

Now, $c_{[\mathcal{L}]}(g^{-1} \cdot x)$ is the scaling factor for $((\widetilde{\varphi}')^{-1})^* \widetilde{e}_{E'}$, and $c_{[\mathcal{L}]}(x)$ is the scaling factor for $(\widetilde{\varphi}^{-1})^* \widetilde{e}_E$. The commutativity of the diagram on the right in (4.4.7.1) gives

$$\widetilde{\varphi}'^{-1} = f \circ \widetilde{\varphi}^{-1} \circ g_p.$$

Combining the effect of pullback via g_p on the canonical pairing and Lemma 4.4.2, we find

$$(\widetilde{\varphi}'^{-1})^* \widetilde{e} = g_p^* (\widetilde{\varphi}^{-1})^* f^* \widetilde{e}_{E'} = g_p^* (\widetilde{\varphi}^{-1})^* \widetilde{e}_E^{\deg f} = g_p^* \left((\widetilde{\varphi}^{-1})^* \widetilde{e}_E \right)^{\deg f} = \left((\widetilde{\varphi}^{-1})^* \widetilde{e}_E \right)^{\deg f \cdot \det(g_p)}. \tag{4.4.7.3}$$

Thus, plugging in the expression (4.4.7.2) for $\deg f$, we find

$$c_{[\mathcal{L}]}(g^{-1} \cdot x) = \det_p(g) \cdot c_{[\mathcal{L}]}(x),$$

giving the claimed equivariance.

The Igusa case can be argued similarly, or deduced from the big Igusa case using equivariance of the projection map (4.3.0.1), Lemma 4.3.3, and commutativity of the diagram (4.4.4.1). $\square$

In particular, Lemma 4.4.7 implies that the fibers of $c_{\text{big Igusa},[\mathcal{L}]}$ (resp. $c_{\text{Igusa},[\mathcal{L}]}$) admit an action of $\ker \det_p$ (resp. $\ker \det_p|_{M_p^\circ \times G^p}$).

4.5. **Projection from Igusa to Katz level and the canonical section.** There is a natural projection map

$$M_{\mathrm{Igusa}} \to M_{\mathrm{Katz}} \tag{4.5.0.1}$$

given by forgetting $\varphi^{\text{ét}}$. This map is a quasi-torsor for the functor

$$\underline{\mathbb{Z}_p^\times} = \mathrm{Aut}(\mathbb{Q}_p/\mathbb{Z}_p)$$

acting on M_{Igusa} by composition with $\varphi^{\text{ét}}$.

In fact, we will now show it is a trivial torsor, and the choice of an equivalence class of lattices $[\mathcal{L}]$ induces a canonical section

$$s_{\mathrm{can},[\mathcal{L}]} : M_{\mathrm{Katz}} \to M_{\mathrm{Igusa}}.$$

Indeed, as in 4.4.3, for $x \in M_{\mathrm{Katz}}(R)$, we have Weil pairings $e_{p^n,x,[\mathcal{L}]}$ on $E_x[p^n]$ and we may trivialize $E[p^n]/\widehat{E}[p^n]$ by sending the generator $1/p^n$ of $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$ to the unique element x of $E[p^n]/\widehat{E}[p^n]$ such that

$$e_{p^n}(\widehat{\varphi}(\cdot), x) : \mu_{p^n} \to \mu_{p^n}$$

is the identity map. These trivializations compile to a trivialization

$$\varphi^{\text{ét}} : E[p^\infty]/\widehat{E} \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p,$$

and thus we obtain our section $s_{\mathrm{can},[\mathcal{L}]}$.

Lemma 4.5.1. *The section $s_{\mathrm{can},[\mathcal{L}]}$ induces an isomorphism*

$$M_{\mathrm{Katz}} \xrightarrow{\sim} c_{[\mathcal{L}]}^{-1}(1).$$

It is $\underline{\mathbb{Z}_p^\times} \times G^p$-equivariant when M_{Katz} is equipped with the natural action and M_{Igusa} is equipped with the action given by

$$\underline{\mathbb{Z}_p^\times} \times G^p \hookrightarrow \ker \det_p \subset M_p^\circ \times G^p \tag{4.5.1.1}$$

$$(a, g^p) \mapsto \left(\left(a, a^{-1}\det_{\mathrm{ur}}^{-1}(g^p)\right), g_p\right) \tag{4.5.1.2}$$

and the natural action of $\ker \det_p$.

Proof. The first claim, that $s_{\mathrm{can},[\mathcal{L}]}$ induces an isomorphism $M_{\mathrm{Katz}} \xrightarrow{\sim} c_{[\mathcal{L}]}^{-1}(1)$ is clear from the constructions; the inverse is given by the restriction to $c_{[\mathcal{L}]}^{-1}(1)$ of the projection map 4.5.0.1. By Lemma 4.4.7, the component $c_{[\mathcal{L}]}^{-1}(1)$ admits an action of the kernel of $\det_p$ in $M_p^\circ \times G^p$. On the other hand, the projection map satisfies the obvious equivariance. Thus, because the map (4.5.1.1) identifies $\underline{\mathbb{Z}_p^\times} \times G^p$ with the kernel of $\det_p$ in $M_p^\circ \times G^p$, we conclude that the inverse map to $s_{\mathrm{can},[\mathcal{L}]}$ satisfies the given equivariance, and thus so does $s_{\mathrm{can},[\mathcal{L}]}$. □

4.6. **Mod π moduli interpretations.**

4.6.1. *Big Igusa level.* Because quasi-isogenies lift uniquely along nilpotent thickenings, we find

Lemma 4.6.2. *For $R \in \mathrm{Nilp}_{\mathbb{Z}_p}$, and π nilpotent in R, the natural reduction map*

$$M_{\text{big Igusa}}(R) \to M_{\text{big Igusa}}(R/\pi)$$

is a bijection.

In particular, if we take $\pi = p$, we find that $M_{\text{big Igusa}}$ is the pull-back to $\text{Nilp}^{\text{op}}_{\mathbb{Z}_p}$ of a moduli problem on $\text{Sch}/\mathbb{F}_p$. Indeed, as observed in [1], $M_{\text{big Igusa}}$ is represented by the Witt vectors of a ring representing $M_{\text{big Igusa},\mathbb{F}_p}$ on $\text{Sch}/\mathbb{F}_p$; we will return to this point in 4.8.2.

4.6.3. *Igusa level.* Let A be a p-adically complete ring and let $\pi \in A$ be topologically nilpotent for the p-adic topology on A. We consider the moduli problem $M_{\text{Igusa}-\pi}$ which classifies for $\text{Spec} R \in \text{Nilp}^{\text{op}}_A$, the set of quadruples

$$(E_0, \mathcal{E}, \psi, \alpha)$$

where $E_0/(R/\pi)$ is an elliptic curve, α is an isomorphism of Zariski sheaves on $\text{Spec} R/\pi$

$$\alpha : \underline{(\mathbb{A}_f^{(p)})^2} \xrightarrow{\sim} V_{\mathbb{A}_f^{(p)}} E,$$

$\mathcal{E}$ is an extension of p-divisible groups over R

$$\mathcal{E} : \widehat{\mathbb{G}_m} \to G_{\mathcal{E}} \to \mathbb{Q}_p/\mathbb{Z}_p,$$

and $\psi : E_0[p^\infty] \xrightarrow{\sim} G_{\mathcal{E}}|_{R/\pi}$, all considered up to isomorphism of $\mathcal{E}$ and quasi-prime-to-p-isogeny of E_0.

There is a natural map $M_{\text{Igusa},A} \to M_{\text{Igusa}-\pi}$: it sends a point in $M_{\text{Igusa},A}(R)$ represented by a quadruple $(E, \widehat{\varphi}, \varphi^{\text{ét}}, \alpha)$ to

$$(E_{R/\pi}, \mathcal{E}_{E[p^\infty], \widehat{\varphi}, \varphi^{\text{ét}}}, \psi_{\text{can}}, \alpha|_{R/\pi}) \in M_{\text{Igusa}-\pi}(R),$$

where ψ_{can} is the canonical isomorphism

$$E_{R/\pi}[p^\infty] = E[p^\infty]_{R/\pi}.$$

As a simple consequence of Serre-Tate lifting theory, we obtain:

Lemma 4.6.4. *The map*

$$M_{\text{Igusa},A} \to M_{\text{Igusa}-\pi}$$

described above is an isomorphism.

4.7. **Representability and moduli problems with finite prime-to-p level.** If we fix a $\widehat{\mathbb{Z}}^{(p)}$-lattice $\mathcal{L}$ in $(\mathbb{A}_f^{(p)})^2$, we may form the variants $M_{*,\mathcal{L}}$ of our previous moduli problems where the prime-to-p level structure is an isomorphism

$$\alpha_{\mathcal{L}} : \underline{\mathcal{L}} \xrightarrow{\sim} T_{\widehat{\mathbb{Z}}^{(p)}} E$$

and the tuples are considered up to isomorphism of the E. It follows from the discussion in 4.1.5 that the natural map $M_{*,\mathcal{L}} \to M_*$ is an equivalence, equivariant for the action of $\underline{\text{GL}(\mathcal{L})} \subset G^p$ and the group action at p (by B_p, M_p, or $\underline{\mathbb{Z}_p}^\times$ as $* = \text{big Igusa, Igusa, or Katz}$)

If we fix also a prime-to-p integer N, we may form the finite prime-to-p level variants where the prime-to-p level structure is an isomorphism

$$\alpha_{\mathcal{L}/N} : \underline{\mathcal{L}/N} \xrightarrow{\sim} E[N]$$

and the tuples are considered up to isomorphism of the elliptic curve. We then have

$$M_{*,\mathcal{L}} = \lim_{(N,p)=1} M_{*,\mathcal{L}/N}.$$

Thus, we may build off of representability at finite prime-to-p level to obtain representability at full prime-to-p level. More precisely, we obtain

Theorem 4.7.1. *For* $*$ *equal to* Katz, Igusa, *or* big Igusa, *the moduli problem* M_* *is representable by a* p*-adically complete ring* $\mathbb{V}_*$*, flat over* $\mathbb{Z}_p$*. Moreover, for any compact open*

$$K^p \subset \mathrm{GL}_2(\mathbb{A}_f^{(p)}),$$

the natural map

$$M_* \to \mathrm{Spf}\mathbb{V}_*^{K^p}$$

is an fpqc $\underline{K^p}$*-torsor; in particular,*

$$\mathrm{Spf}\mathbb{V}_*^{K^p} = M_*/\underline{K^p}.$$

Furthermore, if K^p *stabilizes a lattice* $\mathcal{L}$*,*

$$K_N = \ker \mathrm{GL}(\mathcal{L}) \to \ker \mathrm{GL}(\mathcal{L}/N\mathcal{L}),$$

and $K^p \supset K_N$*, then there is a natural isomorphism*

$$M_{*,\mathcal{L}}/\underline{K^p} = M_{*,\mathcal{L}/N}/(\underline{K^p/K_N}).$$

Proof. As shown by Katz [7], the moduli problem $M_{\mathrm{Katz},\mathcal{L}/N}$ is representable by a p-adically complete ring $\mathbb{V}_{\mathrm{Katz},\mathcal{L}/N}$, flat over $\mathbb{Z}_p$. It is the p-adic completion of the colimit of p-adically complete rings representing the finite level version of the moduli problem parameterizing arithmetic $\Gamma_1(p^n)$ structure at p, i.e., where $\widehat{\varphi}$ is replaced with an injection $\mu_{p^n} \to E[p^n]$.

We claim that

$$\mathbb{V}_{\mathrm{Katz},\mathcal{L}}^{K_N} = \mathbb{V}_{\mathrm{Katz},\mathcal{L}/N}.$$

Indeed, this is true mod p^n for every n because the covers are Galois finite étale, and passes to the completion because we also have $\mathbb{V}_{\mathrm{Katz},\mathcal{L}/N} \hookrightarrow \mathbb{V}_{\mathrm{Katz}}^{K_N}$.

On the other hand, it is more or less clear that

$$M_{\mathrm{Katz},\mathcal{L}/N} = M_{\mathrm{Katz},\mathcal{L}}$$

is a $\underline{K_N}$-torsor: two points in $M_{\mathrm{Katz},\mathcal{L}}$ with the same image in $M_{\mathrm{Katz},\mathcal{L}/N}$ differ by a unique element of $\underline{K_N}$, and the map $M_{\mathrm{Katz},\mathcal{L}} \to M_{\mathrm{Katz},\mathcal{L}/N}$ is an fpqc cover (as a limit of finite étale covers) and thus surjective in the fpqc topology.

Thus for $* =$ Katz we have verified all claims for $K^p = K_N$. Now, a general K^p contains some K_N as a normal subgroup with finite index, and

$$M_{\mathrm{Katz},\mathcal{L}}/K^p = (M_{\mathrm{Katz},\mathcal{L}}/K_N)/(K^p/K_N)$$

so that we are reduced to studying the quotient of the Galois finite étale cover

$$M_{\mathrm{Katz},\mathcal{L}/N} \to M_{\mathrm{Katz},\mathcal{L}/1}$$

by the action of the subgroup $K^p/K_N \subset \mathrm{GL}(\mathcal{L}/N\mathcal{L})$, and we conclude the full result for $* =$ Katz.

We have seen that M_{Igusa} is a trivializable $\underline{\mathbb{Z}_p^\times}$ torsor over M_{Katz}. The choice of a trivialization (e.g. by choosing a lattice $[\mathcal{L}]$ and taking the canonical section $s_{\mathrm{can},[\mathcal{L}]}$) induces an isomorphism

$$M_{\mathrm{Igusa}} \xrightarrow{\sim} \mathrm{Spf}\mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{V}_{\mathrm{Katz}}).$$

The same logic applies also for the prime-to-p level structures $\mathcal{L}/N$, and thus the rest of the proof for Igusa level structure proceeds as the proof for Katz level structure.

Finally, to treat the case of big Igusa level structure, let

$$(E_{\mathrm{univ}}, \widehat{\varphi}_{\mathrm{univ}}, \widehat{\varphi}_{\mathrm{univ}}^{\mathrm{\acute{e}t}})$$

be the universal elliptic curve with Igusa level structure over M_{Igusa}. We find that M is the limit of finite flat covers parameterizing splittings of $\mathcal{E}_{E_{\mathrm{univ}}[p^\infty],\widehat{\varphi}_{\mathrm{univ}},\widehat{\varphi}^{\text{ét}}}$ (cf. the proof of Lemma 4.3.3), and thus representable by the p-adic completion of the colimit of these rings. Again the same logic also applies for the finite level prime-to-p level structures $\mathcal{L}/N$, and the rest of the proof again follows as before. □

Example 4.7.2. If $\Gamma_1(N)$ denotes the subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}}^{(p)})$ congruent to

$$\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \mod N$$

then we find $\mathrm{Spf}\mathbb{V}_{\mathrm{Katz}}^{\Gamma_1(N)}$ represents the moduli problem classifying triples $(E, \widehat{\varphi}, P)$ where E is an elliptic curve, $\widehat{\varphi} : \widehat{E} \xrightarrow{\sim} \widehat{\mathbb{G}_m}$, and P is a point of exact order N on E, considered up to isomorphism of E.

4.8. **The Frobenius.** We consider now the action of the diagonal quasi-isogeny $(p^n, 1) \subset M_p \subset B_p$ on the moduli problem $M_{\mathrm{big\,Igusa}}$, and induced maps on M_{Igusa} and M_{Katz}.

Let $x \in M_{\mathrm{big\,Igusa}}(R)$ and fix a distinguished representative $(E, \widetilde{\varphi}, \alpha)$ for x as in 4.1.5 where φ is induced by an isomorphism φ. Via φ, we may identify μ_{p^n} as a subgroup of E.

There is a unique isomorphism φ' making the following diagram commute:

$$\begin{tikzcd} E \arrow[rr] & & E/\mu_{p^n} \\ E[p^\infty] \arrow[u] \arrow[r] \arrow[d, "\varphi"] & E[p^\infty]/\mu_{p^n} = & E/\mu_{p^n}[p^\infty] \arrow[u] \arrow[d, "\varphi'"] \\ \widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p \arrow[rr, "(p^n,1)"] & & \widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p \end{tikzcd}$$

Thus, we find that

$$(p^n, 1) \cdot x = (E/\mu_{p^n}, \widetilde{\varphi'}, \alpha')$$

where the prime-to-p level structure α' on E/μ_{p^n} is induced from the prime-to-p level structure α on E by pushforward through the isogeny $E \to E/\mu_{p^n}$.

4.8.1. *Exotic isomorphisms.* Now,

$$(p^n, 1) \cdot T_p\widehat{\mathbb{G}_m} \cdot (p^n, 1)^{-1} = p^n T_p\widehat{\mathbb{G}_m},$$

and thus we obtain an isomorphism

$$(p^n, 1) : M_{\mathrm{Igusa}} = T_p\widehat{\mathbb{G}_m}\backslash M_{G_{\mathrm{can}}} \xrightarrow{\sim} (p^n T_p\widehat{\mathbb{G}_m})\backslash M_{G_{\mathrm{can}}},$$

and, combined with the projection to M_{Igusa}, a map

$$\mathrm{Frob}^n_{\mathrm{can}} : M_{\mathrm{Igusa}} \to M_{\mathrm{Igusa}}.$$

Applying the computation from the previous paragraph, we find that

$$\mathrm{Frob}^n_{\mathrm{can}}\left((E, \widehat{\varphi}, \varphi^{\text{ét}})\right) = (E/\mu_{p^n}, \widehat{\varphi}', (\varphi^{\text{ét}})'),$$

where $\widehat{\varphi}'$ and $(\varphi^{\text{ét}})'$ are obtained in the obvious way. In particular, $\mathrm{Frob}^n_{\mathrm{can}}$ is a lift of the p^n-power frobenius on $M_{\mathrm{Igusa},\mathbb{F}_p}$.

We can identify $(p^n T_p \widehat{\mathbb{G}_m})\backslash M_{G_{\mathrm{can}}}$ as the moduli problem

$$M_{\mathrm{Igusa,split}\,p^n}/M_{\mathrm{Igusa}}$$

whose fiber over $x \in M_{\mathrm{Igusa}}(R)$ parameterizes splittings of the p^n-torsion extension $\mathcal{E}_x[p^n]$. The isomorphism $M_{\mathrm{Igusa}} \xrightarrow{\sim} M_{\mathrm{Igusa,split}\,p^n}$ induced by $(p^n, 1)$ is given by remembering the canonical splitting

$$\frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \xrightarrow{(\varphi^{\text{ét}})^{-1}} E[p^n]/\mu_{p^n} \hookrightarrow (E/\mu_{p^n})[p^n].$$

Similarly, viewing

$$M_{\mathrm{Katz}} = \left(T_p\widehat{\mathbb{G}_m} \rtimes (1 \times \mathbb{Z}_p^\times)\right)\backslash M_{\mathrm{Igusa}},$$

and defining the relative moduli problem

$$M_{\mathrm{Katz,split}\,p^n}/M_{\mathrm{Katz}}$$

parameterizing splittings of

$$\mu_{p^n} \to E[p^n] \to E[p^n]/\mu_{p^n},$$

we obtain Frobenius lifts $\mathrm{Frob}^n_{\mathrm{can}}$ from the action of $(p^n, 1)$, which factor as isomorphisms

$$M_{\mathrm{Katz}} \xrightarrow{\sim} M_{\mathrm{Katz,split}\,p^n} \tag{4.8.1.1}$$

followed by the natural projection. The isomorphisms (4.8.1.1) are the exotic isomorphisms of [8, Lemma 5.6.3].

4.8.2. *Frobenius and representability.* It is clear from the definitions that

$$M_{\mathrm{big\,Igusa}} = \lim_n M_{\mathrm{Igusa,split}\,p^n}.$$

Rewriting this limit using the exotic isomorphisms, we find

$$M_{\mathrm{big\,Igusa}} = \lim_{\mathrm{Frob_{can}}} M_{\mathrm{Igusa}}.$$

As observed by Caraiani-Scholze [1], this implies that $\mathbb{V}_{\mathrm{big\,Igusa}}$ is isomorphic to the Witt vectors of the perfection of $\mathbb{V}_{\mathrm{Igusa},\mathbb{F}_p}$. Combined with our description of $\mathbb{V}_{\mathrm{Igusa}}$, we find

$$\mathbb{V}_{\mathrm{big\,Igusa}} \cong \mathrm{Cont}(\mathbb{Z}_p^\times, W(\mathbb{V}^{\mathrm{perf}}_{\mathrm{Katz},\mathbb{F}_p})).$$

4.9. **p-adic modular forms.** We now explain Katz's embedding of modular forms in $\mathbb{V}_{\mathrm{Katz}}$. Let $E_{\mathrm{univ}}/M_{\mathrm{Katz}}$ denote the universal elliptic curve up to prime-to-p-isogeny.

We have a natural trivialization of $\omega = \omega_{E_{\mathrm{univ}}}$ given by $\widehat{\varphi}^*_{\mathrm{univ}}\frac{dt}{t}$ and the isomorphism $\omega_{E_{\mathrm{univ}}} = \omega_{\widehat{E_{\mathrm{univ}}}}$. This trivialization is fixed by the G^p-action and transforms by the inverse of the identity character for the $\underline{\mathbb{Z}_p^\times}$-action, thus we obtain an isomorphism of equivariant line bundles

$$\omega^k \xrightarrow{\sim} \mathcal{O} \otimes z^{-k} \boxtimes \mathrm{triv}$$

where $z^{-k} \boxtimes \mathrm{triv}$ is the character of

$$\underline{\mathbb{Z}_p^\times} \times G^p$$

which is z^{-k} on the first component and trivial on the second.

There is an equivariant Kodaira-Spencer isomorphism

$$\omega^2 \xrightarrow{\sim} \Omega \otimes H^2_{\mathrm{dR}}(E_{\mathrm{univ}}). \tag{4.9.0.1}$$

For an elliptic curve, the trace gives a canonical trivialization of $H^2_{\mathrm{dR}}(E_{\mathrm{univ}})$, but this is not preserved by isogeny – instead it is multiplied by the degree. Thus, we find that the choice of an equivalence class of lattices $[\mathcal{L}]$ induces a canonical trivialization of $H^2_{\mathrm{dR}}(E)$ on M_{Katz}, but it is not equivariant for the action of G^p. Instead, it introduces a twist by the unramified determinant character $\det_{\mathrm{ur}}$.

Thus the choice of an equivalence class of lattices $[\mathcal{L}]$ induces an equivariant isomorphism

$$H^2_{\mathrm{dR}}(E) \xrightarrow{\sim} \mathcal{O} \otimes \det_{\mathrm{ur}},$$

and combining this with the Kodaira-spencer map, we obtain an equivariant isomorphism

$$\omega_E^2 \xrightarrow{\sim} \Omega \otimes \det_{\mathrm{ur}}.$$

Using the canonical trivialization of ω_E, we thus obtain an isomorphism of $\underline{\mathbb{Z}_p}^\times \times G^p$-equivariant bundles

$$\omega_E^{k-2} \otimes \Omega \xrightarrow{\sim} \mathcal{O} \otimes z^{-k} \boxtimes \det_{\mathrm{ur}}^{-1}. \tag{4.9.0.2}$$

Now, there is a map from M_{Katz} to the formal p-adic completion $Y^\wedge_{\mathrm{ord}}$ of the ordinary locus Y_{ord} on the classical modular curve over $\mathbb{Z}_{(p)}$

$$Y := \lim_{K^p} Y_{\mathrm{GL}_2(\mathbb{Z}_p)K^p}.$$

The map is given by forgetting $\widehat{\varphi}$. This realizes M_{Katz} as a $\underline{\mathbb{Z}_p^\times}$-torsor over this locus. In particular, $H^0(Y^\wedge_{\mathrm{ord}}, \omega_E^{k-2} \otimes \Omega)$ is identified $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariantly with the $\mathbb{Z}_p^\times$-invariant elements in $H^0(Y^\wedge_{\mathrm{ord}}, \omega_E^{k-2} \otimes \Omega)$. Via the trivialization (4.9.0.2), these are identified $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariantly with the elements of $\mathbb{V}_{\mathrm{Katz}} \otimes \det_{\mathrm{ur}}^{-1}$ which transform via the character z^k under the $\mathbb{Z}_p^\times$-action.

Extending this, one defines p-adic modular forms of weight κ for any continuous character κ with values in a p-adically complete ring R as the G^p representation

$$\mathbb{V}_{\mathrm{Katz},R}[\kappa] \otimes \det_{\mathrm{ur}}^{-1}.$$

On K^p-invariants, the action of $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$ induces an action of the abstract prime-to-p double coset Hecke algebra

$$R[K^p \backslash \mathrm{GL}_2(\mathbb{A}_f^{(p)})/K^p]$$

preserving the space of weight κ p-adic modular forms of level K^p (i.e. the K^p-invariants), and for integral weights this definition matches the Hecke action given classically when viewing a modular form as a section of a line bundle on $Y^\wedge_{\mathrm{ord},\mathrm{GL}_2(\mathbb{Z}_p)K^p}$.

5. The $\widehat{\mathbb{G}_m}$-action

5.1. **Extending the action on M_{Igusa}.** We have seen that M_{Igusa} admits a natural action of $M_p^\circ \times G^p$, described in terms of the moduli problem. Using the presentation

$$M_{\mathrm{Igusa}} = N_p^\circ \backslash M_{\mathrm{big\,Igusa}}$$

given by Lemma 4.3.3, we will enlarge this to an action of a larger group.

Let B'_p be the pre-image of M_p° in B_p under the natural projection; it contains N_p and M_p° under the natural inclusions and is a semi-direct product

$$B'_p = N_p \rtimes M_p^\circ \subset B_p = N_p \rtimes M_p.$$

Then, N_p° is a normal subgroup in B'_p, and, using Lemma 2.3.1 for $\widehat{\mathbb{G}_m}$, we find that there is an fpqc quotient

$$B'_p/N_p^\circ = \overline{B_p} := \widehat{\mathbb{G}_m} \rtimes M_p.$$

Because $M_{\mathrm{big\,Igusa}}$ admits an action of $B_p \times G^p$, and in particular of the subgroup $B'_p \times G^p$, we find that M_{Igusa} admits an action of $\overline{B_p} \times G^p$, and the equivariance of the projection map shows that this action extends the moduli action of $M_p^\circ \times G^p$. The new part of the action is that of $\widehat{\mathbb{G}_m} = N_p/N_p^\circ$; the statement that we have an action of $\overline{B_p} \times G^p$ includes the existence of this action plus a compatibility with the moduli action of $M_p^\circ \times G^p$.

5.2. **Extending the action on M_{Katz}.** The map $\det_p$ restricted to $B'_p \times G^p$ factors through $\overline{B_p} \times G^p$, and we write $\overline{B_p}^1$ for the kernel in $\overline{B_p} \times G^p$ of the induced map.

The group $\widehat{\mathbb{G}_m} = \widehat{\mathbb{G}_m} \times 1$ is a normal subgroup of $\overline{B_p}^1$. We also have a natural inclusion of $\underline{\mathbb{Z}_p}^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ in $\overline{B_p}^1$ induced by the map (4.5.1.1) of Lemma 4.5.1, and together these give a presentation

$$\overline{B_p}^1 = \widehat{\mathbb{G}_m} \rtimes (\underline{\mathbb{Z}_p}^\times \times G^p)$$

where the conjugation action of $\underline{\mathbb{Z}_p}^\times \times G^p$ on $\widehat{\mathbb{G}_m}$ is given by

$$(a, g^p) \cdot \zeta \cdot (a, g^p)^{-1} = \zeta^{a^2 \det_{\mathrm{ur}}(g^p)}.$$

If we fix an equivalence class $[\mathcal{L}]$ of $\widehat{\mathbb{Z}}^{(p)}$ lattices in $(\mathbb{A}_f^{(p)})^2$, then by Lemma 4.5.1 the section $s_{\mathrm{can},[\mathcal{L}]}$ induces an isomorphism

$$M_{\mathrm{Katz}} \xrightarrow{\sim} c^{-1}_{\mathrm{Igusa},[\mathcal{L}]}(1)$$

which is equivariant for the action of $\underline{\mathbb{Z}_p}^\times \times G^p$. On the other hand, from the equivariance of the p-component maps (Lemma 4.4.7) and commutativity of (4.4.4.1), we find that the action of $\overline{B_p}$ on M_{Igusa} constructed in the previous section induces an action of $\overline{B_p}^1$ on $c^{-1}_{\mathrm{Igusa},[\mathcal{L}]}(1)$.

Thus, we have extended the action of $\underline{\mathbb{Z}_p}^\times \times G^p$ to an action of

$$\widehat{\mathbb{G}_m} \rtimes (\underline{\mathbb{Z}_p}^\times \times G^p).$$

In particular, taking $[\mathcal{L}] = [(\widehat{\mathbb{Z}}^{(p)})^2]$ and restricting and restricting to $\widehat{\mathbb{G}_m}$, we obtain a $\widehat{\mathbb{G}_m}$-action that will turn out to be the action described in Theorem A. Moreover, we have established the second part of Theorem A for this action.

Remark 5.2.1. For two lattice classes $[\mathcal{L}]$ and $[\mathcal{L}']$, the induced $\widehat{\mathbb{G}_m}$-actions differ by the automorphism $[\mathcal{L} : \mathcal{L}'] \in \mathbb{Z}_p^\times$ of $\widehat{\mathbb{G}_m}$.

The fact that different components lead to different actions is a reflection of the fact that we cannot obtain a $\widehat{\mathbb{G}_m}$ action on M_{Katz} using the presentation

$$M_{\mathrm{Katz}} = (N_p^\circ \rtimes (1 \times \underline{\mathbb{Z}_p}^\times)) \backslash M_{\mathrm{big\,Igusa}}$$

because $N_p^\circ \rtimes (1 \times \underline{\mathbb{Z}_p}^\times)$ is not a normal subgroup of $N_p \rtimes (1 \times \underline{\mathbb{Z}_p}^\times)$.

5.3. **Differentiating the action.** We consider the action map

$$\widehat{\mathbb{G}_m} \times M_{\mathrm{Igusa}} \to M_{\mathrm{Igusa}},$$

describing the action of $\widehat{\mathbb{G}_m} \subset \overline{B_p}$ on M_{Igusa}. To differentiate it, we compose with the tangent vector $t\partial_t$ at the identity in $\widehat{\mathbb{G}_m}$. The latter is given by a map $D \to \mathbb{G}_m$ which in coordinates is

$$\mathbb{Z}_p[t^{\pm 1}] \to \mathbb{Z}_p[\epsilon]/\epsilon^2,\ t \mapsto 1+\epsilon.$$

Thus, the composition of the action map with $t\partial_t$ gives a vector field on M_{Igusa} described as a map

$$t_{\mathrm{u}} : D \times M_{\mathrm{Igusa}} \to M_{\mathrm{Igusa}}.$$

On the other hand, we have the universal extension

$$\mathcal{E}_{E_{\mathrm{univ}}[p^\infty], \widehat{\varphi}_{\mathrm{univ}}, \varphi^{\text{ét}}_{\mathrm{univ}}} : \widehat{\mathbb{G}_m} \to E_{\mathrm{univ}}[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p$$

(cf. (4.1.4.1)), and, as explained in 3.1, this extension gives rise to a differential $d\tau \in \Omega_{M_{\mathrm{Igusa}}}$. We will show:

Theorem 5.3.1. *Notation as above,*

$$d\tau(t_{\mathrm{u}}) = 1$$

Because the section

$$s_{\mathrm{can},[(\widehat{\mathbb{Z}}^{(p)})^2]} : M_{\mathrm{Katz}} \to M_{\mathrm{Igusa}}$$

identifies the $\widehat{\mathbb{G}_m}$ action on M_{Igusa} with the $\widehat{\mathbb{G}_m}$ action on M_{Katz} and pulls back the universal extension to the universal extension on M_{Katz}, Theorem 5.3.1 implies the first part of Theorem A from the introduction. The second part of Theorem A having already been shown in the previous section, this will complete our proof of Theorem A.

Proof of Theorem 5.3.1. It suffices to work over $\mathbb{Z}/p^n$ for arbitrary n. We abbreviate $S = M_{\mathrm{Igusa}}|_{\mathbb{Z}/p^n}$ and $R = \mathbb{V}_{\mathrm{Igusa}}/p^n$ so that $S = \mathrm{Spec} R$. We write $\pi : E \to S$ for the universal elliptic curve up-to-prime-to-p-isogeny over S and $\widehat{\varphi}$, $\varphi^{\text{ét}}$ for the universal trivializations of $\widehat{E}$ and $E[p^\infty]/\widehat{E}$.

We recall the definition of $d\tau$ (mod p^n): we have the canonical extension

$$\mathcal{E}_{E[p^\infty], \widehat{\varphi}, \varphi^{\text{ét}}} : \widehat{\mathbb{G}_m} \to E[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p$$

and the induced trivialization ω_{can}, u_{can} of the crystal the vector bundle $\mathrm{Lie} EE[p^\infty]$ with connection ∇_{crys}, and $d\tau$ is defined by the equation

$$\nabla_{\mathrm{crys}}(\omega_{\mathrm{can}}) = u_{\mathrm{can}} d\tau..$$

As in 2.6.2, we write $\nabla_{\mathrm{crys}, t_{\mathrm{u}}}$ for the isomorphism

$$t_{\mathrm{u}}^* \mathrm{Lie} EE[p^\infty] \xrightarrow{\sim} 0^* \mathrm{Lie} EE[p^\infty]$$

over $D \times S$ induced by ∇_{crys}. In light of (2.6.2.1), it suffices to show that

$$\nabla_{\mathrm{crys}, t_{\mathrm{u}}}(\omega_{\mathrm{can}}) = \omega_{\mathrm{can}} + \epsilon \cdot u_{\mathrm{can}}. \tag{5.3.1.1}$$

Now, we have that $t_{\mathrm{u}} = (1+\epsilon) \cdot 0$ via the $\widehat{\mathbb{G}_m}$ action, where we view the tangent vectors t_{u} and 0 as $R[\epsilon]$-points of S and $(1+\epsilon)$ as an $R[\epsilon]$-point of $\widehat{\mathbb{G}_m}$. Thus, if we write

$$(E_0, \widehat{\varphi}_0, \varphi_0^{\text{ét}}, \alpha_0)$$

for the quadruple classified by 0 and

$$(E_{\mathrm{u}}, \widehat{\varphi}_{\mathrm{u}}, \varphi_{\mathrm{u}}^{\text{ét}}, \alpha_{\mathrm{u}})$$

for the triple classified by t_{u}, we have

$$(1+\epsilon)\cdot(E_0, \widehat{\varphi}_0, \varphi_0^{\text{ét}}, \alpha_0) = (E_{\mathrm{u}}, \widehat{\varphi}_{\mathrm{u}}, \varphi_{\mathrm{u}}^{\text{ét}}, \alpha_{\mathrm{u}}) \tag{5.3.1.2}$$

In particular, $\nabla_{\mathrm{crys},t_{\mathrm{u}}}$ is identified with the Messing isomorphism

$$\mathrm{Lie}EE_{\mathrm{u}}[p^\infty]^\vee \xrightarrow{\sim} \mathrm{Lie}EE_0[p^\infty]^\vee$$

induced by the isomorphism

$$E_{\mathrm{u}} \mod \epsilon = E_0 \mod \epsilon$$

given by $1+\epsilon = 1 \mod \epsilon$ and (5.3.1.2).

To compute this, we pass to the flat cover

$$S_\infty = \mathrm{Spec}\mathbb{V}_{\text{big Igusa}}/p^n[\epsilon, (1+\epsilon)^{1/p^\infty}]/\epsilon^2.$$

Over $\mathbb{V}_{\text{big Igusa}}/p^n[\epsilon]$ and thus over S_∞, we have a canonical splitting of $\mathcal{E}_{E_0[p^\infty], \widehat{\varphi}_0, \varphi^{\text{ét}}}$ which gives an isomorphism

$$\varphi_0 : E_0[p^\infty]_{R_\infty} \xrightarrow{\sim} \widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p.$$

If we let

$$g_\epsilon := \begin{pmatrix} 1 & (1+\epsilon, (1+\epsilon)^{1/p}, \ldots) \\ 0 & 1 \end{pmatrix} \in N_p(S_\infty)$$

our description of the unipotent action in 4.2.3 then shows that over S_∞, E_{u} is the Serre-Tate lift to S_∞ corresponding to the isomorphism

$$g_\epsilon \circ \varphi_0 : E_0|_{S_\infty/\epsilon}[p^\infty] \to \widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p.$$

Thus, the Messing isomorphism in the canonical basis is identified over S_∞ with the map

$$\mathrm{Lie}E(\mathbb{Q}_p/\mathbb{Z}_p \times \widehat{\mathbb{G}_m}) \to \mathrm{Lie}E(\mathbb{Q}_p/\mathbb{Z}_p \times \widehat{\mathbb{G}_m})$$

induced by g_ϵ^t. If we write this in the canonical basis we get a map

$$\mathbb{G}_a \frac{dt}{t} \times \mathbb{G}_a t\partial_t \to \mathbb{G}_a \frac{dt}{t} \times \mathbb{G}_a t\partial_t,$$

and, by Lemma 2.5.1, it is given by

$$\begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}.$$

By construction, these bases are identified with the bases $\omega_{\mathrm{can}}, u_{\mathrm{can}}$, and thus we obtain equation (5.3.1.1), concluding the proof.

□

5.4. **Computing the action.** Recall that for π-topologically nilpotent in R we defined in 4.6.3 a moduli problem $M_{\mathrm{Igusa}-\pi}$ canonically isomorphic to $M_{\mathrm{Igusa},R}$ which places the emphasis on the canonical extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\widehat{\mathbb{G}_m}$. Recall also from 3.2 that there is a Kummer construction which, given $q \in \mathbb{G}_m(R)$ produces an extension of p-divisible groups

$$\mathcal{E}_q : 0 \to \widehat{\mathbb{G}_m} \to G_q \to \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Using the explicit description of the unipotent action in 4.2.3, we find

Theorem 5.4.1. *Suppose $\zeta \in \widehat{\mathbb{G}_m}(R)$ and $\pi \in R$ is such that $\zeta \equiv 1 \mod \pi$, and*

$$(E_0, \mathcal{E}_q, \psi, \alpha_N) \in M_{\mathrm{Igusa}-\pi}(R)$$

for $q \in \mathbb{G}_m(R)$. Then

$$\zeta \cdot (E_0, \mathcal{E}_q, \psi, \alpha) = (E_0, \mathcal{E}_{\zeta^{-1}q}, \psi', \alpha)$$

where ψ' is the composition of ψ with the canonical identification

$$\mathcal{E}_q|_{R/\pi} = \mathcal{E}_{\zeta^{-1}q}|_{R/\pi}$$

coming from $q \equiv \zeta^{-1}q \mod \pi$.

Proof. If we write x_1 for the point

$$(E_0, \mathcal{E}_q, \psi, \alpha_N) \in M_{\mathrm{Igusa}}(R)$$

and x_2 for the point

$$\zeta \cdot (E_0, \mathcal{E}_q, \psi, \alpha_N) = (E_0, \mathcal{E}_{\zeta^{-1}q}, \psi', \alpha_N)$$

It suffices to show that over the cover $R[q^{1/p^\infty}, \zeta^{1/p^\infty}]$, there are lifts of these points to $M_{\text{big Igusa}}$ and a lift $\tilde{\zeta}$ of ζ to $\widetilde{\widehat{\mathbb{G}_m}}$ such that $\tilde{\zeta} \cdot \tilde{x}_1 = \tilde{x}_2$. The desired lifts are given by the splittings $1/p^n \mapsto (q^{1/p^n}, 1/p^n)$ and $1/p^n \mapsto \zeta^{-1/p^n} q^{1/p^n}$ of $\mathcal{E}_q$ and $\mathcal{E}_{\zeta^{-1}q}$, respectively, and $\tilde{\zeta} = (\zeta^{1/p^n})_n$. That $\tilde{\zeta} \cdot \tilde{x}_1 = \tilde{x}_2$ then follows from commutativity of the following diagram mod $\zeta - 1$:

$$\begin{tikzcd}
G_q \arrow[r, "="] & G_{\zeta^{-1}q} \\
\widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p \arrow[u, "1/p^n \mapsto (q^{1/p^n}, 1/p^n)"] \arrow[r, "\begin{pmatrix} 1 & \tilde{\zeta} \\ 0 & 1 \end{pmatrix}"'] & \widehat{\mathbb{G}_m} \times \mathbb{Q}_p/\mathbb{Z}_p \arrow[u, "1/p^n \mapsto (\zeta^{-1/p^n} q^{1/p^n}, 1/p^n)"']
\end{tikzcd}$$

□

5.4.2. *Action on the Tate curve and q-expansions.* Let

$$R = \left(\mathrm{colim}_{(N,p)=1} W(\overline{\mathbb{F}_p})((q^{1/N}))\right)^\wedge$$

and consider the Tate curve $\mathrm{Tate}(q)$ over R. We have the canonical trivialization

$$\varphi_{\mathrm{can}} : \widehat{\mathrm{Tate}(q)} \xrightarrow{\sim} \widehat{\mathbb{G}_m},$$

and, if we fix a compatible system of prime-to-p roots of unity ζ_N, we obtain a basis $(\zeta_N)_N, (q^{1/N})_N$ for $T_{\widehat{\mathbb{Z}}^{(p)}}\mathrm{Tate}(q)$ over R and thus a trivialization $\alpha_{(\zeta_N)_N,(q^{1/N})_N}$ of the prime-to-p adelic Tate module.

The *cusps* of M_{Katz} are the R-points in the $GL_2(\widehat{\mathbb{Z}}^{(p)})$-orbit of

$$\left(\mathrm{Tate}(q), \widehat{\varphi}_{\mathrm{can}}, \alpha_{(\zeta_N)_N,(q^{1/N})_N}\right).$$

For $g \in \mathbb{V}_{\mathrm{Katz},A}$ and a cusp c, we call the element $c(g) \in A\widehat{\otimes}R$ the q-expansion of g at c. We find

Corollary 5.4.3. *If c is a cusp of M_{Katz} and $g \in \mathbb{V}_{\mathrm{Katz},A}$ has q-expansion at c*

$$\sum_{k\in\mathbb{Z}_{(p)}} a_k q^k,$$

then, for $\zeta \in \widehat{\mathbb{G}_m}(R)$,

$$\zeta \cdot g := (\zeta^{-1}\cdot)^* g$$

has q-expansion at c

$$\sum_{k\in\mathbb{Z}_{(p)}} a_k(\zeta q)^k = \sum_{k\in\mathbb{Z}_{(p)}} \zeta^k a_k q^k \tag{5.4.3.1}$$

(where the powers ζ^k make sense because $k \in \mathbb{Z}_{(p)} \subset \mathbb{Z}_p^\times$).

Proof. The q-expansion in 5.4.3.1 is the image of $c(g)$ under the the map

$$\begin{aligned}\gamma_\zeta : R\widehat{\otimes}A &\to R\widehat{\otimes}A\\ q^k &\mapsto (\zeta q)^k.\end{aligned}$$

Thus, because the action of $\widehat{\mathbb{G}_m}$ commutes with the action of $\mathrm{GL}_2(\widehat{\mathbb{Z}}^{(p)})$ (because the latter is in the kernel of $\det_{\mathrm{ur}}$), we may assume our cusp is given by the triple

$$\left(\mathrm{Tate}(q), \widehat{\varphi}_{\mathrm{can}}, \alpha_{(\zeta_N)_N,(q^{1/N})_N}\right).$$

Then, it follows from Theorem 5.4.1 that

$$\zeta^{-1} \cdot \Big(\mathrm{Tate}(q), \widehat{\varphi}_{\mathrm{can}}, (\zeta_N, q^{1/N})\Big) = \Big(\mathrm{Tate}(\zeta q), \widehat{\varphi}_{\mathrm{can}}, (\zeta_N, \zeta^{1/N} q^{1/n}\Big)).$$

This is the base change of

$$\left(\mathrm{Tate}(q), \widehat{\varphi}_{\mathrm{can}}, \alpha_{(\zeta_N)_N,(q^{1/N})_N}\right)$$

through γ_ζ, and thus we conclude. □

Remark 5.4.4. Using Corollary 5.4.3 over $A = \mathbb{Z}_p[\epsilon]$, we find that if we differentiate the $\widehat{\mathbb{G}_m}$-action along $t\partial_t$ in the sense of 5.3, the induced operator on q-expansions is $-q\partial_q = -\theta$ (we get a minus sign because to get the derivation in 5.3 we did not compose with an inverse as we have to obtain the natural left action on functions).

Remark 5.4.5. In this remark we show that for E_{univ} the universal curve over $M_{\mathrm{Katz},\overline{\mathbb{F}_p}}$, the canonical extension $\mathcal{E}_{E_{\mathrm{univ}},\widehat{\varphi}_{\mathrm{univ}}}$ is not a Kummer p-divisible group (in the sense of 3.2). In particular, this shows Theorem 5.4.1 cannot be applied directly to compute the action on all points of M_{Katz}.

Suppose by way of contradiction that $\mathcal{E}_{E_{\mathrm{univ}},\widehat{\varphi}_{\mathrm{univ}}}$ were of the form $\mathcal{E}_{q_{\mathrm{univ}}}$ for $q_{\mathrm{univ}} \in \mathbb{V}_{\mathrm{Katz},\overline{\mathbb{F}_p}}$. Then, writing $M_{\mathrm{Katz},\overline{\mathbb{F}_p}}$ as a limit of finite level moduli problems (cf. 4.7) parameterizing inclusions we find that q_{univ} only depends on $\widehat{\varphi}|_{\mu_{p^n}}$ and the $\underline{K^p}$ orbit of α for some $n > 0$ and some compact open $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^{(p)})$. In particular, we find that the element $1 + p^n \in \mathbb{Z}_p^\times$ fixes q_{univ}.

Because

$$\left(\mathrm{Tate}(q), (1+p^n) \circ \widehat{\varphi}_{\mathrm{can}}, \alpha_{(\zeta_N, q^{1/N})}\right) = 1 + p^n \cdot \left(\mathrm{Tate}(q), \widehat{\varphi}_{\mathrm{can}}, \alpha_{(\zeta_N, q^{1/N})}\right),$$

we find that q_{univ} of these two points agree. Because

$$\mathcal{E}_{\mathrm{Tate}(q), \widehat{\varphi}_{\mathrm{can}}} = \mathcal{E}_q$$

and

$$\mathcal{E}_{\mathrm{Tate}(q), (1+p^n) \circ \widehat{\varphi}_{\mathrm{can}}} = \mathcal{E}_{q^{2p^n+2}},$$

we conclude $\mathcal{E}_{q^{2p^n+2}}$ is isomorphic to $\mathcal{E}_q$ over R. By Lemma 3.2.7 this cannot be the case, giving a contradiction.

Note that we could make a similar argument using the formal neighborhood of an ordinary point instead of the Tate curve.

5.4.6. *Action on the formal neighborhood of an ordinary curve.* We may also consider the local expansions at an ordinary point: to do so, fix an elliptic curve $E_0/\overline{\mathbb{F}_p}$, and consider its universal deformation $E/\mathrm{Spf} R$ where R is a smooth complete 2-dimensional local ring over $W(\overline{\mathbb{F}_p})$ with residue field $\overline{\mathbb{F}_p}$. If we fix trivializations

$$\widehat{\varphi}_0 : \widehat{E_0} \xrightarrow{\sim} \widehat{\mathbb{G}_m} \text{ and } \varphi^{\text{ét}} : E_0[p^\infty]/\widehat{E} \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p,$$

these deform uniquely to a trivializations

$$\widehat{\varphi}_0 : \widehat{E} \xrightarrow{\sim} \widehat{\mathbb{G}_m} \text{ and } \varphi^{\text{ét}} : E[p^\infty]/\widehat{E} \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p.$$

We thus obtain a map $\mathrm{Spf} R \to M_{\mathrm{Igusa}}$.

Remark 5.4.7. The choice of a basis $x \in T_p E_0(\overline{\mathbb{F}_p})$ is equivalent the to the choice $\varphi_0^{\text{ét}}$, and via the Weil pairing, also gives a choice of $\widehat{\varphi}_0$, $(\bullet, x) : \widehat{E_0} \to \widehat{\mathbb{G}_m}$.

The extension

$$\mathcal{E}_{E[p^\infty], \widehat{\varphi}, \widehat{\varphi^{\text{ét}}}} : \widehat{\mathbb{G}_m} \to E[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p$$

is of the form G_q for a unique $q \in \widehat{\mathbb{G}_m}(R)$, and, as in Remark 3.2.6, the element q^{-1} is the Serre-Tate coordinate for this extension. From Theorem 5.4.1, we deduce that the action of $\widehat{\mathbb{G}_m}$ preserves the formal subscheme $\mathrm{Spf} R \subset M_{\mathrm{Katz}}$, and on this formal subscheme ζ acts as multiplication by ζ on the Serre-Tate coordinate q^{-1}.

5.5. **The action of** $\mathrm{Cont}(\mathbb{Z}_p, R)$**.** For any p-adically complete ring R, the action map for the $\widehat{\mathbb{G}_m}$-action gives a continuous map

$$a^* : \mathbb{V}_{\mathrm{Katz},R} \to R[[T]] \widehat{\otimes}_R \mathbb{V}_{\mathrm{Katz},R}.$$

The natural left action of $\widehat{\mathbb{G}_m}(R)$ on $\mathbb{V}_{\mathrm{Katz},R}$ is by $\zeta \cdot g = (\zeta^{-1} \cdot)^* g$, and we can express this using the action map: if we consider $\zeta \in \widehat{\mathbb{G}_m}(R)$ as the map $R[[T]] \to R$ given by $T \mapsto \zeta$, and write ι for the inverse map $\widehat{\mathbb{G}_m} \to \widehat{\mathbb{G}_m}$, then $\zeta \cdot g$ is the image of

$$(\iota \times \mathrm{Id})^* a^* g$$

under the induced map

$$\zeta_{\mathbb{V}_{\mathrm{Katz},R}} : R[[T]] \widehat{\otimes}_R \mathbb{V}_{\mathrm{Katz},R} \to \mathbb{V}_{\mathrm{Katz},R}.$$

More generally, if we identify $R[[T]]$ with the continuous R-linear dual of $\mathrm{Cont}(\mathbb{Z}_p, R)$ via the Amice transform, we obtain an R-linear map

$$\begin{aligned}\mathrm{Cont}(\mathbb{Z}_p, R) \times \mathbb{V}_{\mathrm{Katz},R} &\to \mathbb{V}_{\mathrm{Katz},R}\\ (f, g) &\mapsto f \cdot g := \langle f, (\iota \times \mathrm{Id})^* a^* g\rangle.\end{aligned}$$

If we let χ_ζ be the R-valued character of $\mathbb{Z}_p$ given by $\chi_\zeta(a) = \zeta^a$, viewed as an element of $\mathrm{Cont}(\mathbb{Z}_p, R)$, we find

$$\chi_\zeta \cdot g = \zeta \cdot g.$$

That we have an action of $\widehat{\mathbb{G}_m}$ to begin with is equivalent to this being an algebra action of $\mathrm{Cont}(\mathbb{Z}_p, R)$ on $\mathbb{V}_{\mathrm{Katz},R}$.

More generally, we find:

Theorem 5.5.1. *If $g \in \mathbb{V}_{\mathrm{Katz},R}$ has q-expansion at a cusp c*

$$c(g) = \sum_{k\in\mathbb{Z}_{(p)}} a_k q^k,$$

then the q-expansion of $f \cdot g$ is

$$c(f \cdot g) = \sum_{k\in\mathbb{Z}_{(p)}} f(k) a_k q^k.$$

Proof. Because for a general R,

$$\mathbb{V}_{\mathrm{Katz},R} = \mathbb{V}_{\mathrm{Katz},\mathbb{Z}_p} \widehat{\otimes} R \text{ and } \mathrm{Cont}(\mathbb{Z}_p, R) = \mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)\widehat{\otimes} R,$$

it suffices to verify this for $R = \mathbb{Z}_p$. Then, because the base change is injective on the target ring for q-expansions, it suffices work over $R = \mathcal{O}_{\mathbb{C}_p}$. Moreover it suffices to verify the identity for the action of locally constant functions on $\mathbb{Z}_p$, which are dense in continuous functions. But for any locally constant function, some $\mathcal{O}_{\mathbb{C}_p}$-multiple can be written as a linear combination of characters, and thus Corollary 5.4.3 gives the result for a multiple of each locally constant function. Since the total q-expansion map (i.e. the product of the q-expansion maps over all cusps) is injective and the target ring is torsion-free over $\mathcal{O}_{\mathbb{C}_p}$, we find that the result holds for all locally constant functions. □

5.6. **A classical construction and Gouvea's twisting measure.** In [4, III.6.2], Gouvea constructed the algebra action of the previous section on $\Gamma_1(N)\backslash M_{\mathrm{Katz}}$ (viewed as a "twisting measure"). The construction of Gouvea has the advantage of using only classical notions in the p-adic theory of modular forms, available already in work of Katz [9], but does not provide the same conceptual clarity as the modular construction from the isogeny action on the big Igusa variety.

Gouvea's construction can be paraphrased in our language as follows: by the q-expansion principle, it suffices to construct the action of finite order characters. This action can be obtained as the natural action of μ_{p^n} on $M_{\mathrm{Katz,split}\,p^n} \subset M_{\mathrm{Igusa,split}\,p^n}$ (via the canonical section $s_{\mathrm{can},[(\widehat{\mathbb{Z}}^{(p)})^2]}$), viewed as an action on M_{Katz} through the exotic isomorphisms of 4.8.1. This can be computed explicitly for the Tate curve, and we obtain the claimed action.

This construction can be modified slightly to give the $\widehat{\mathbb{G}_m}$-action directly instead of the algebra action: once we have the actions of μ_{p^n} on M_{Katz}, the computation on q-expansions and the q-expansion principle shows that they are compatible for varying n, and thus compile to an action of $\widehat{\mathbb{G}_m}$. That this action agrees with the $\widehat{\mathbb{G}_m}$-action as constructed previously follows from the considerations of 4.8.1.

6. Eisenstein measures

In a series of papers ([7, 9, 8]), Katz introduced increasingly general *Eisenstein measures* with values in $\mathbb{V}_{\mathrm{Katz}}$ interpolating Eisenstein series. These Eisenstein measures specialize at the cusps and ordinary CM points to p-adic L-functions interpolating L-values of Dirichlet characters (at the cusps) and grossencharacter (at ordinary CM points).

The papers [7, 9] are concerned with single variable p-adic L-functions, whereas [8] gives two variable L-functions by interpolating not just holomorphic Eisenstein series but also certain real analytic Eisenstein series.

In this section, we explain how "half" of the two-variable measure can be produced by a type of convolution of the single-variable measure with the $\widehat{\mathbb{G}_m}$-action. To keep the exposition clear, we work at level $K^p = \mathrm{GL}_2(\widehat{\mathbb{Z}}^{(p)})$ away from p.

Remark 6.0.1. The real analytic Eisenstein series are related to the holomorphic Eisenstein series by iterated application of the differential operator θ – thus, we can summarize the difference between our approach and that of Katz by saying that instead of applying θ and then interpolating, we have first interpolated θ and then applied this interpolated operator to the holomorphic Eisenstein measure.

6.1. **Measures.**

For R a p-adically complete $\mathbb{Z}_p$-algebra and X a profinite set, an R-valued measure on X is an element

$$\mu \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Cont}(X, \mathbb{Z}_p), R).$$

Note that such a μ is automatically continuous for the p-adic topology on $\mathrm{Cont}(X, \mathbb{Z}_p)$ and R. In fact, the stronger basic congruence property holds: if $f \equiv g \mod p^n$, then $\mu(f) \equiv \mu(g) \mod p^n$ – this observation is at the heart of the application of measures to p-adic L functions.

Remark 6.1.1. An R-valued distribution is an R-valued functional on the space of locally constant functions on X, $C^\infty(X, \mathbb{Z}_p)$. The space $C^\infty(X, \mathbb{Z}_p)$ is dense in $\mathrm{Cont}(X, \mathbb{Z}_p)$, thus when R is p-adically complete a distribution automatically completes to a measure, and the two notions are equivalent. We will use this below.

6.1.2. *Measures on products.*

Proposition 6.1.3. *Let X and Y be profinite sets, and R a p-adically complete $\mathbb{Z}_p$-algebra. If $(\,,\,)$ is an R-valued $\mathbb{Z}_p$-bilinear pairing on* $\mathrm{Cont}(X, \mathbb{Z}_p) \times \mathrm{Cont}(Y, \mathbb{Z}_p)$, *then there is a unique R-valued measure μ on $X \times Y$ such that for $f \in$* $\mathrm{Cont}(X, \mathbb{Z}_p)$ *and $g \in$* $\mathrm{Cont}(Y, \mathbb{Z}_p)$,

$$\mu(fg) = (f, g). \tag{6.1.3.1}$$

Proof. By Remark 6.1.1, it suffices to construct a functional on $C^\infty(X \times Y, \mathbb{Z}_p)$ satisfying (6.1.3.1), and then verify that (6.1.3.1) holds for any continuous f and g and the unique extension of that distribution to a measure. So, suppose we have constructed a measure μ such that (6.1.3.1) holds for f and g locally constant.

Then, for any continuous f and g and $n \in \mathbb{Z}_{>0}$, pick f_n and g_n locally constant such that $f \equiv f_n \mod p^n$ and $g \equiv g_n \mod p^n$. Then

$$fg \equiv f_n g_n \mod p^n$$

and thus

$$\mu(fg) \equiv \mu(f_n g_n) \equiv (f_n, g_n) \equiv (f, g) \mod p^n$$

and we conclude $\mu(fg) = (f, g)$.

Thus it remains to construct the distribution and show that it is unique. The bilinear pairing $(\ ,\ )$ induces a functional on $C^\infty(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C^\infty(Y, \mathbb{Z}_p)$, thus to conclude, it suffices to show that the product map

$$C^\infty(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C^\infty(Y, \mathbb{Z}_p) \to C^\infty(X \times Y, \mathbb{Z}_p)$$

is an isomorphism: For any profinite set W, $C^\infty(W, \mathbb{Z}_p)$ is the colimit over finite coverings $\mathcal{U} = \{U_1, ..., U_n\}$ of W by disjoint compact opens of $C^\infty_U(W, \mathbb{Z}_p)$, the space of functions constant on each of the U_i. In particular, if $\mathcal{U} = \{U_1, ..., U_n\}$ is such a cover of X and $\mathcal{V} = \{V_1, ..., V_m\}$ is such a cover of Y then

$$\mathcal{U} \times \mathcal{V} := \{U_i \times V_j\}$$

is such a cover of $X \times Y$, and the covers of this form are cofinal for covers of $X \times Y$ by disjoint compact opens. Considering the basis of characteristic functions, we find that the product map induces an isomorphism

$$C^\infty_{\mathcal{U}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C^\infty_{\mathcal{V}}(Y, \mathbb{Z}_p) \to C^\infty_{\mathcal{U} \times \mathcal{V}}(X \times Y, \mathbb{Z}_p)$$

and passing to the colimit over covers $\mathcal{U}$ and $\mathcal{V}$, we conclude. $\square$

Example 6.1.4. If ν_X and ν_Y are R-valued measures on X and Y, respectively, then

$$(f, g) \mapsto \nu_X(f)\nu_Y(g)$$

is a bilinear form and the resulting measure μ on $X \times Y$ is the product measure.

6.2. **Katz's Eisenstein measures.** In this section, we write $\mathbb{V} = \mathbb{V}_{\mathrm{Katz}}^{\mathrm{GL}_2(\widehat{\mathbb{Z}}^{(p)})}$ for the ring representing the Katz moduli problem with no prime-to-p level structure.

6.2.1. *Single variable measures.* In [7, XII], Katz introduced the single variable Eisenstein measures

$$\mu^{(a)} : \mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathbb{V}$$

characterized by the moments

$$\mu^{(a)}(z^{k-1}) = (1 - a^k) 2G_k$$

where $2G_k$ for $k \geq 2$ is the Eisenstein series with q-expansion

$$\zeta(1-k) + 2\sum_{n=1}^{\infty} q^n \cdot \sum_{d|n} d^{k-1}$$

and $2G_1 = 0$.

Remark 6.2.2. For f a locally constant function on $\mathbb{Z}_p$, $\mu^{(a)}$ satisfies the following additional interpolation property [9, Corollary 3.3.8][2]:

$$\mu^{(a)}(z^{k-1}) = (1 - a^k) 2G_{k,f}$$

[2] In this reference, $\mu^{(a)} = H^{a,1}$ for $N = 1$, except for a shift from k to $k + 1$.

where $2G_{k,f}$ has q-expansion

$$L(1-k,f)+2\sum_{n=1}^{\infty}q^n\cdot\sum_{d|n}d^{k-1}.$$

6.2.3. *Two variable measures.* In [8], Katz introduced the two variable Eisenstein-Ramanujan measures

$$\mu^{(a,1)}:\mathrm{Cont}(\mathbb{Z}_p\times\mathbb{Z}_p,\mathbb{Z}_p)\to\mathbb{V}$$

characterized by the moments

$$\mu^{(a,1)}(x^ky^r)=(1-a^{k+r+1})\Phi_{k,r}$$

where

$$\Phi_{k,r}=\begin{cases}2G_{k+r+1} & \text{if } k=0 \text{ or } r=0\\ 2\sum_{n=1}^{\infty}q^n\sum_{dd'=n}d^kd'^r & \text{if } k,r\neq 0\end{cases}$$

In other words,

$$\Phi_{k,r}=\begin{cases}\theta^r2G_{k+1-r} & \text{if } k\geq r\\ \theta^k2G_{r+1-k} & \text{if } r\geq k\end{cases}$$

Remark 6.2.4. The symmetry between $k\geq r$ and $r\leq k$ becomes more complicated as soon as we consider locally constant functions as in Remark 6.2.2.

6.2.5. *Halving the measure.* Our technique will only recover "half" of the measure, i.e. only the moments for $k=\geq r$. To make this precise, consider the map

$$\varphi:\mathbb{Z}_p\times\mathbb{Z}_p\to\mathbb{Z}_p\times\mathbb{Z}_p$$
$$(x,y)\mapsto(x,xy)$$

with image the subset of (x,y) with $|y|\leq|x|$. The measure $\varphi_*\mu^{(a,1)}$ is characterized by the moments

$$\varphi_*\mu^{(a,1)}(x^sy^t)=\mu^{(a,1)}(x^{s+t}y^t)=(1-a^{s+2t+1})\theta^t2G_{s+1} \tag{6.2.5.1}$$

6.3. Convolution of the one-variable measure and the action map.

Theorem 6.3.1. *There is a $\mathbb{V}$-valued measure ν on $\mathbb{Z}_p\times\mathbb{Z}_p$ with moments*

$$\nu(x^sy^t)=\psi((1-a^{s+1})2G_{s+1})(y^t)=(1-a^{s+1})\theta^t2G_{s+1}.$$

Proof. From the one-variable Eisenstein measure $\mu^{(a)}$ and the action map

$$\psi:\mathbb{V}\to\mathrm{Meas}(\mathbb{Z}_p,\mathbb{V}),$$

we obtain a bilinear form on $\mathrm{Cont}(\mathbb{Z}_p,\mathbb{Z}_p)\times\mathrm{Cont}(\mathbb{Z}_p,\mathbb{Z}_p)$

$$(f,g)\mapsto\psi(\mu^{(a)}(f))(g)$$

and thus, by Proposition 6.1.3, a measure ν such that

$$\nu(x^sy^t)=\psi((1-a^{s+1})2G_{s+1})(y^t)=(1-a^{s+1})\theta^t2G_{s+1}.$$

□

Comparing with (6.2.5.1), we see that the measure ν interpolates the same Eisenstein series as $\varphi_*\mu^{(a,1)}$, although with a different normalizing factor (recall that this normalizing factor removes the powers of p in the denominator of the constant term $\zeta(1-k)$ of G_k when $k \equiv -1 \mod p-1$). This type of construction may be useful for studying special values of families of automorphic forms and their images under differential operators on other Shimura varieties where explicit computations with q-expansions are not always available.

7. Ordinary p-adic modular forms

7.1. **Ordinary modular forms.** We consider the sub-ring $\mathbb{V}_{\mathrm{Katz,hol}} \subset \mathbb{V}_{\mathrm{Katz}}$ of functions with holomorphic q-expansions (i.e. q-expansion contained in $W[[q^{1/N}]]$). By the theory of Hida, for any $g \in \mathbb{V}_{\mathrm{Katz,hol}}$, the limit

$$\lim_{n\to\infty} U_p^{n!} f$$

exists, and the assignment sending g to this limit defines an idempotent operator, the ordinary projector

$$e : \mathbb{V}_{\mathrm{Katz,hol}} \to \mathbb{V}_{\mathrm{Katz,hol}}.$$

We define the ordinary part

$$\mathbb{V}^{\mathrm{ord}}_{\mathrm{Katz,hol}} := e\mathbb{V}_{\mathrm{Katz,hol}}$$

so that

$$\mathbb{V}_{\mathrm{Katz,hol}} = \mathbb{V}^{\mathrm{ord}}_{\mathrm{Katz,hol}} \oplus \ker e.$$

7.2. **The sheaf on $\mathbb{Z}_p$ and the fiber at 0.** By the considerations of 5.5, the space $\mathbb{V}_{\mathrm{Katz,hol}}$ is a $\mathbb{Z}_p^\times$-equivariant p-adically complete topological module over $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$. We may thus think of it as a quasi-coherent sheaf $\mathcal{V}$ on the profinite set $\mathbb{Z}_p$, thought of as the formal scheme

$$\mathbb{Z}_p = \mathrm{Spf}\mathrm{Cont}(\mathbb{Z}_p, W).$$

We have an action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p$ where $a \in \mathbb{Z}_p^\times$ acts as multiplication by a^2. The $\mathbb{Z}_p^\times$ action on $\mathbb{V}_{\mathrm{Katz,hol}}$ can be interpreted by saying that the corresponding sheaf $\mathcal{V}$ is $\mathbb{Z}_p^\times$-equivariant. We also have the multiplication by p map $P : \mathbb{Z}_p \to \mathbb{Z}_p$, and the U_p operator induces an isomorphism $U : p^*\mathcal{V} \to \mathcal{V}$.

The point $0 \in \mathbb{Z}_p$ is a fixed-point for both the $\mathbb{Z}_p^\times$-action and multiplication by p, and thus the fiber at 0, $\mathcal{V}|_0$ admits a $\mathbb{Z}_p^\times$-action and is equipped with an automorphism induced by U. The restriction map from global sections to 0 is equivariant for these structures (the endomorphism U_p on global sections is given by restriction to $p\mathbb{Z}_p$ composed with U).

Note that we can interpret this completely algebraically:

$$\mathcal{V}|_0 = \mathbb{V}_{\mathrm{Katz,hol}}/(\mathfrak{m}\mathbb{V}_{\mathrm{Katz,hol}})^\wedge$$

where $\mathfrak{m}$ is the ideal of functions in $\mathrm{Cont}(\mathbb{Z}_p, W)$ vanishing at 0 and $(\cdot)^\wedge$ denotes p-adic completion. The restriction map from global sections is identified with the quotient map, and the $\mathbb{Z}_p^\times$ and U actions are induced on the quotient because $\mathbb{Z}_p^\times$ and U_p both preserve $\mathfrak{m}\mathbb{V}_{\mathrm{Katz,hol}}$.

7.3. **Comparison.** The main result of this section is

Theorem 7.3.1. *The kernel of the restriction map*

$$\mathbb{V}_{\mathrm{Katz,hol}} = \mathcal{V}(\mathbb{Z}_p) \to \mathcal{V}|_0$$

is equal to the kernel of the ordinary projection, $\ker e$*. In particular, restriction induces a* $\mathbb{Z}_p^\times$*,* U_p*-equivariant isomorphism*

$$\mathbb{V}_{\mathrm{Katz}}^{\mathrm{ord}} \xrightarrow{\sim} \mathcal{V}|_0.$$

Proof. As both the kernel of e and the kernel of restriction are closed, it suffices to verify the identity mod p^n. To say that an element $g \in \mathbb{V}_{\mathrm{Katz,hol}}$ is in $\ker e$ is then the same as saying that there is some n such that the coefficient of q^a in its q-expansion is zero whenever $p^n | a$. Certainly the elements of $\mathfrak{m} \cdot \mathbb{V}_{\mathrm{Katz,hol}}$ meet this condition, and on the other hand, if $g \in \ker e$ and n is as above, then $g = f \cdot g$ where f is the locally constant function which is 1 on $\mathbb{Z}_p - p^n\mathbb{Z}_p$ and 0 on $p^n\mathbb{Z}_p$, so $g \in \mathfrak{m} \cdot \mathbb{V}_{\mathrm{Katz,hol}}$. □

7.4. **Questions.** The construction of this section leads to some natural questions, some of which we plan to return to in future work:

(1) Can we explain the finiteness properties of the space of ordinary modular forms shown by Hida from the perspective of the $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$-action?

(2) By looking at the fiber at zero, can we give a proof that the p-adic Banach $\mathrm{GL}_2(\mathbb{Q}_p)$-representations attached to ordinary modular forms admit a non-zero map to principal series that is in the same spirit as looking at the constant term for classical modular forms? In the same vein, what does the restriction away from zero have to do with p-adic Whittaker and Kirillov models?

(3) Is there a reasonable sense in which the whole space $\mathbb{V}_{\mathrm{Katz,hol}}$, admits good finiteness properties over
$$\mathrm{Cont}(\mathbb{Z}_p, W) \rtimes \mathbb{Z}_p[[\mathbb{Z}_p^\times]]?$$

(4) If we consider instead functions on the Igusa variety, then the natural action of
$$\overline{B}_p = \widehat{\mathbb{G}_m} \rtimes (\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)$$
commutes with the natural action of $\mathrm{GL}_2(\mathbb{A}_f)$. Note that the combined action is completely determined by the action on $\mathbb{V}_{\mathrm{Katz}}$ via induction. If we fix a Hecke eigensystem, what information about the corresponding Galois representation is encoded by considering the representation of this algebra on a Hecke eigenspace in $\mathbb{V}_{\mathrm{Katz,hol}}$? What about for the action of B_p on functions on the big Igusa variety? In future work we will explain a connection between this representation and (ϕ, Γ)-modules; the appearance of ordinary modular forms above is a shadow of this connection, related to the fact that ordinary modular forms are precisely those for which D^{nr} of the associated Galois representation is non-zero.

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