

THE UNIVERSITY OF CHICAGO

OVERCONVERGENT MODULAR FORMS AND THE P -ADIC
JACQUET-LANGLANDS CORRESPONDENCE

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS

10 JUNE 2017

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“Music, earthquakes, pie, standing waves, prime number vibrations that don’t exist in the real world, automorphic forms, funk core reality, terrifying math music from beyond the void, mathematicians still use blackboards, p is equal to 11 NOT ℓ ... One might surmise that a madman had taken hostages with a piece of chalk, but some jerk keeps asking follow-up questions, so I’ll assume he’s laying down some solid math knowledge.”

– *Matt Miller (reviewing my defense).*

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ACKNOWLEDGMENTS

I thank Maddie, my wife, who was with me from the start and has never left my side. I thank my family, who have always supported me in all of my pursuits. I thank my friends and fellow graduate students for making it a journey worth taking.

I thank Matt Emerton, my advisor, for his profound influence, not just on this work but on all of my mathematics. I thank all of the other teachers and mentors I have had along the way, who are too many to name but who haven't been forgotten.

I thank Frank Calegari for serving as the second reader of this thesis, and Ngo Bao Chau for serving as my secondary advisor.

During the preparation of this work, I have benefited from helpful conversations with many mathematicians, including: Rebecca Bellovin, Ana Caraiani, Tianqi Fan, David Hansen, Christian Johannson, Kiran Kedlaya, Erick Knight, Daniel Le, Keerthi Madapusi-Pera, Jay Pottharst, Peter Scholze, Matthias Strauch, Jan Vonk, Jared Weinstein, and Yiwen Zhou. In particular, I thank Rebecca Bellovin and Jay Pottharst for pointing out the connection between an early version of Corollary C and the work of Serre [26], Ana Caraiani for explaining some of the results on the Hodge-Tate period map in [4], and Erick Knight for suggesting that an earlier version of Theorem A should be packaged as an isomorphism of Hecke algebras.

ABSTRACT

We construct a global p -adic Jacquet-Langlands transfer from overconvergent modular forms to naive p -adic automorphic forms on the quaternion algebra over \mathbb{Q} ramified at p and ∞ , answering an old question of Serre [26, paragraph (26)]. Using this transfer, we show that the completed Hecke algebra of naive automorphic forms on the quaternion algebra is isomorphic to the completed Hecke algebra of modular forms, and, conditional on a local-global compatibility conjecture, obtain new information about the local p -adic Jacquet-Langlands correspondence of Knight and Scholze. The construction and proofs live entirely in the world of p -adic geometry; in particular we do not use the smooth Jacquet-Langlands correspondence as an input.

CHAPTER 1

INTRODUCTION

1.1 Summary

Let p be a prime number, and let D be the quaternion algebra over \mathbb{Q} ramified at p and ∞ . If we fix a compact open $K^p \subset D^\times(\mathbb{A}_f^{(p)})$, we can form the naive space of p -adic automorphic functions on D^\times of level K^p ,

$$\text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \mathbb{Q}_p). \quad (1.1.0.1)$$

Here the double coset is equipped with its natural topology as a profinite set. For primes l at which the level K^p is a maximal compact, we obtain commuting Hecke operators T_l and S_l acting on this space. We also have an action of $D^\times(\mathbb{Q}_p)$ via right multiplication, and we will be interested in the structure of the space (1.1.0.1) under these symmetries.

We have two main results. The first, Theorem A below, shows that the p -adically completed spectral theory of the Hecke operators acting on this space is equivalent to the p -adically completed spectral theory of the Hecke operators acting on classical modular forms, which in turn is equivalent to the p -adically completed spectral theory of Hecke operators on the completed cohomology of GL_2 . This is a type of spectral p -adic Jacquet-Langlands correspondence.

Our second main result, given in Theorem B and Corollary C below, refines this comparison in the case of overconvergent modular forms. We show that a Hecke eigenform in the space of overconvergent modular forms can be transferred to a Hecke eigenform in the space of p -adic automorphic functions while retaining control over the action of a maximal torus of $D^\times(\mathbb{Q}_p)$. This functoriality result lives properly in the world of the p -adic Langlands program, which is a refinement of the Langlands program that takes into account the richer structure of p -adic representations of the Galois group of \mathbb{Q}_p and the p -adic representation

theory of p -adic groups.

We highlight that this result applies to *any* overconvergent eigenform – in particular, we do not require the eigenform to be finite slope (as is common in results on overconvergent modular forms) or any discrete series condition at p (as one might expect from the smooth Jacquet-Langlands correspondence). In particular, the result applies to *all* classical modular forms, including those which are principal series at p .

These results answer an old question of Serre [26, paragraph (26)] by generalizing the mod p Jacquet-Langlands of [26] to p -adic modular forms. We also conjecture a local-global compatibility statement with the local p -adic Jacquet-Langlands correspondences of Knight [19] and Scholze [22], under which our construction gives important information about the structure of the quaternionic representations appearing in this correspondence (cf. Corollary 8.0.4).

The strategy of Serre in the mod p case is to realize a finite quaternionic double coset as a subset of the special fiber of a modular curve over which the modular sheaf ω has a natural trivialization. To go from modular forms to quaternionic functions, one simply evaluates on this trivialization; in the other direction, one uses the Hasse invariant and ampleness of ω in order to extend functions off of the super-singular locus. We follow a similar strategy, enhanced by modern developments in the p -adic geometry of modular curves: we realize the full quaternionic double coset as a closed profinite subset of the infinite level modular curve and evaluate (overconvergent) modular forms on a natural trivialization over this set. In the other direction, we extend functions off of this subset using the technique of fake Hasse invariants as in [24].

The Hodge-Tate period map of [24] (along with some refinements in [4]) plays a fundamental role in this work, both at a conceptual and technical level. In particular, along the way to proving Theorem B, we give a construction of overconvergent modular forms which uses the the Hodge-Tate period map to reduce to the study of equivariant bundles on \mathbb{P}^1 , where the geometry is simple to understand. This construction generalizes naturally to other

Shimura varieties, as does our approach to explicit functorialities in the p -adic Langlands program; these generalizations will appear in a later work.

1.2 Statement of results

1.2.1 Hecke algebras.

To state our results, we will first fix an isomorphism

$$D_{\mathbb{A}_f^{(p)}}^\times \cong \mathrm{GL}_{2, \mathbb{A}_f^{(p)}}. \quad (1.2.1.1)$$

Via the isomorphism (1.2.1.1) we can consider K^p as a subgroup of $\mathrm{GL}_2(\mathbb{A}_F^{(p)})$.

It will be convenient to choose a concrete realization of D and this isomorphism. Let E_0 be a supersingular elliptic curve over $\overline{\mathbb{F}_p}$ and let $D = \mathrm{End}(E_0) \otimes \mathbb{Q}$. We also fix a basis for the prime-to- p Tate module

$$T_{\widehat{\mathbb{Z}}^{(p)}} E_0 := \varprojlim_{(n,p)=1} E_0[n](\overline{\mathbb{F}_p}).$$

Then, the isomorphism (1.2.1.1) is obtained via the action of endomorphisms on $T_{\widehat{\mathbb{Z}}^{(p)}} E_0$ in this basis.

We consider the abstract Hecke algebra of level K^p

$$\mathbb{T}_{\mathrm{abs}} = \mathbb{Z}_p[D^\times(\mathbb{A}_f^{(p)})//K^p] = \mathbb{Z}_p[\mathrm{GL}_2(\mathbb{A}_f^{(p)})//K^p]$$

where the second equality comes from our fixed isomorphism.

Our p -adic Banach space of quaternionic automorphic forms (1.1.0.1) admits an action of $\mathbb{T}_{\mathrm{abs}}$. We will consider two other p -adic Banach spaces equipped with actions of $\mathbb{T}_{\mathrm{abs}}$: the space $\mathbb{V} \otimes \mathbb{Q}_p$ of Katz p -adic modular functions of level K^p as in [16], and the completed cohomology \widehat{H}^1 of the tower of modular curve at prime-to- p level K^p [10].

There are action maps

$$\begin{aligned} \mathbb{T}_{\text{abs}} &\rightarrow \text{End}(\text{Cont}(D^\times(\mathbb{Q}_p)\backslash D^\times(\mathbb{A}_f)/K^p, \mathbb{Q}_p)), \\ \mathbb{T}_{\text{abs}} &\rightarrow \text{End}(\mathbb{V} \otimes \mathbb{Q}_p), \\ \text{and } \mathbb{T}_{\text{abs}} &\rightarrow \text{End}(\widehat{H}^1). \end{aligned}$$

We denote by \mathbb{T}_{D^\times} , \mathbb{T}_{mf} , and \mathbb{T}_{GL_2} , respectively, the completions of the images of \mathbb{T}_{abs} in each of these spaces with respect to the topology of pointwise convergence.

Theorem A. *The topological \mathbb{T}_{abs} -algebras \mathbb{T}_{mf} , \mathbb{T}_{GL_2} and \mathbb{T}_{D^\times} are isomorphic.*

Remark 1.2.2. Any such isomorphism is unique, as the image of \mathbb{T}_{abs} is dense in each space.

Remark 1.2.3. The theorem also holds if \mathbb{T}_{abs} is replaced with any \mathbb{Z}_p sub-algebra in the formation of the completed Hecke algebras and in the statement of the theorem. For example, one can consider the sub-algebra generated by the commuting Hecke operators at primes l where K^p is maximal compact.

It is well-known that \mathbb{T}_{mf} and \mathbb{T}_{GL_2} are isomorphic (cf. [9]), thus the new content of Theorem A is the isomorphism with \mathbb{T}_{D^\times} . We note that our proof of Theorem A lives fully in the world of p -adic geometry, and does not pass through the classical Jacquet-Langlands correspondence for locally algebraic vectors in the space of quaternionic automorphic forms. In particular, Theorem A provides an alternative p -adic proof of the existence of Galois representations attached to quaternionic automorphic forms.

1.2.4 Overconvergent modular forms.

Let F/\mathbb{Q}_p be a quadratic extension. We define \check{F} to be the completion of the maximal unramified extension of F and \check{F}_∞ to be the compositum of \check{F} and the (non-complete)

Lubin-Tate extension of F . Let E/\check{F} be a complete extension, and let

$$\kappa \in \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, E^\times)$$

We will refer to κ as the weight. The Lie algebra of \mathbb{Z}_p^\times is spanned by the derivative of the identity character, and we write $\text{Lie}\kappa \in E$ for $d\kappa$ expressed in the dual basis for $(\text{Lie}\mathbb{Z}_p^\times)^*$.

For a fixed radius of overconvergence w^1 , we will define a space M_κ^w of w -overconvergent modular forms of weight κ and tame level K^p equipped with an action of \mathbb{T}_{abs} . These will contain the spaces of overconvergent forms considered, e.g., by Pilloni [21], however in our setup it is natural to use larger spaces which include all possible levels at p .

We will fix an embedding $F \hookrightarrow D(\mathbb{Q}_p)$ corresponding to a CM lift of $E_0[p^\infty]$ and consider the action of F^\times on

$$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p$$

through this embedding. Under this embedding, F^\times is a maximal torus inside $D^\times(\mathbb{Q}_p)$. In the next two paragraphs we introduce some language to describe actions of F^\times .

Let τ be the identity character of F^\times acting on F and $\bar{\tau}$ its conjugate. If we consider F^\times as a Lie group over \mathbb{Q}_p , then its Lie algebra $\text{Lie}F^\times$ is two dimensional, and after base change to F , $d\tau$ and $d\bar{\tau}$ are a basis for the space of characters $(\text{Lie}F^\times)^*$. For L an extension of F and $a, b \in L$ we write $L[a, b]$ for the one-dimensional vector space L equipped with the action of $\text{Lie}F^\times$ by $ad\tau + bd\bar{\tau}$.

Given a representation of F^\times on a Banach space V , we denote by $V^{F^\times-\text{an}}$ the vectors which are locally analytic for the action of F^\times (viewed as a two dimensional Lie group over \mathbb{Q}_p). The space $V^{F^\times-\text{an}}$ admits an action of $\text{Lie}F^\times$.

1. For us, it will be natural to define the radius of overconvergence using the Hodge-Tate period map rather than the Hasse invariant – in either case we obtain a decreasing system of neighborhoods with intersection equal to the closure of the ordinary locus, and so the difference is irrelevant when considering all overconvergent modular forms.

Theorem B. *There is a \mathbb{T}_{abs} -equivariant embedding*

$$M_{\kappa}^w \hookrightarrow \text{Cont}(D^{\times}(\mathbb{Q}_p) \backslash D^{\times}(\mathbb{A}_f) / K^p, E \cdot \check{F}_{\infty})^{F^{\times} - \text{an}}$$

It factors through the $E \cdot \check{F}_{\infty}[\text{Lie}\kappa - 1, -1]$ -isotypic component for the action of $\text{Lie}F^{\times}$.

Remark 1.2.5. The embedding is not unique; in fact, for any fixed F, k , and w we will construct many such embeddings, corresponding roughly to different points in $\mathbb{P}^1(F) \backslash \mathbb{P}^1(\mathbb{Q}_p)$ contained in a neighborhood of $\infty \in \mathbb{P}^1(\mathbb{Q}_p)$ depending on w . If we fix such a choice, and a level K_p at p , then we can replace \check{F}_{∞} with a finite Lubin-Tate extension \check{F}_n .

In particular, we deduce

Corollary C. *If g is an overconvergent modular form over a discretely valued $E \subset \widehat{F}$ of weight $\kappa \in E$ and g is a simultaneous eigenvector for some sub-algebra of \mathbb{T}_{abs} , then there exists a simultaneous eigenvector with the same eigenvalues*

$$f \in \text{Cont}(D^{\times}(\mathbb{Q}_p) \backslash D^{\times}(\mathbb{A}_f) / K^p, E)^{F^{\times} - \text{an}}$$

contained in the $E[\text{Lie}\kappa - 1, -1]$ -isotypic component.

The control over the action of the maximal torus $F^{\times} \subset D^{\times}(\mathbb{Q}_p)$ is important in applications to the p-adic Langlands program. For example, if our conjectural local-global compatibility, Conjecture 8.0.1, holds, then Corollary C implies that for local representations of $D^{\times}(\mathbb{Q}_p)$ arising via restriction from global automorphic representations, the locally algebraic vectors are not dense (Corollary 8.0.4).

1.3 Related work

As discussed earlier in the introduction, Serre [26] proved mod p analogs of Theorems A and B, and part of our work, suitably interpreted, is a characteristic zero lift of Serre's construction (cf. 5.8.4).

Emerton [11, 3.3.2] proved a version of Theorem A after localizing at a maximal ideal of the Hecke algebra (under some minor restrictions on the residual representation). His proof uses the classical Jacquet-Langlands correspondence and deep results in the deformation theory of Galois representations. We note that our proof of Theorem A uses *none* of these tools – it lives entirely within the world of p -adic geometry.

Knight [19] and Scholze [22] have both produced *local* p -adic Jacquet-Langlands correspondences. These correspondences satisfy local-global compatibility with the completed cohomology of Shimura curves, and Chojecki-Knight [7] have announced a proof via patching that the two correspondences agree. As discussed above, we conjecture that our construction satisfies a local-global compatibility statement with this local correspondence (cf. Section 8).

In Section 7 we give a construction of overconvergent modular forms by working at infinite level. Chojecki-Hansen-Johannson gave an equivalent construction of overconvergent modular forms for Shimura curves over \mathbb{Q} in [6], and applied it to study the overconvergent Eichler-Shimura isomorphism of Andreatta-Iovita-Stevens [1]. The emphasis in the presentation is different, however, the key ideas in the constructions are the same.

There has also been considerable work done on the global p -adic Jacquet-Langlands correspondence for definite quaternion algebras over \mathbb{Q} which are unramified at p – cf. e.g. [5, 20]. The flavor of the p -adic Jacquet-Langlands correspondence when p is unramified is different from the ramified case we study, as in the unramified case only the p -adic representation theory of $GL_2(\mathbb{Q}_p)$ is involved, which is better understood than that of $D^\times(\mathbb{Q}_p)$.

1.4 Generalizations

In order to control the scope of this document and highlight the connections with Serre’s mod p Jacquet-Langlands [26], we work only with GL_2/\mathbb{Q} . However, much of this work generalizes naturally to other groups admitting Shimura varieties (under some natural hypotheses on the existence of perfectoid infinite level Shimura varieties and Hodge-Tate period maps, which are now known in many cases). For example, all of our results generalize immediately

if GL_2 is replaced by the units in any quaternion algebra Q/\mathbb{Q} split at infinity and p and D is replaced with the quaternion algebra ramified at the places where Q is as well as at p and ∞ . These generalizations will be the topic of future work of the author.

More generally, the ideas of this paper can be used to compare overconvergent automorphic forms and certain completed cohomology groups related to Shimura varieties and Igusa varieties. The basic philosophy is that overconvergent automorphic forms (and some variants), through the Hodge-Tate period map, mediate a comparison between locally analytic representations constructed from the geometry of flag varieties and the Hodge-Tate weight 0 part of completed cohomology groups. This leads to interesting consequences already for the completed H^1 of modular curves; some results in this direction will appear in future work of the author.

1.5 Outline

In Chapter 2 we cover geometric preliminaries. We recall some subtleties of working with perfectoid spaces over discretely valued fields, and aspects of equivariant geometry that will be useful in organizing our constructions. We also discuss the realizations of profinite sets as formal schemes and adic spaces.

In Chapter 3 we recall the classical constructions of automorphic bundles over modular curves, and then explain the perfectoid construction via the Hodge-Tate period map due to Scholze [24] (cf. also [4]). Because we want to work over \mathbb{Q}_p rather than a perfectoid extension, some care is necessary in keeping track of group actions while unraveling a Tate twist – our main contribution in this section is to give a careful $\mathrm{GL}_2(\mathbb{A}_f)$ -equivariant description of the Hodge-Tate filtration on the perfectoid modular curve over \mathbb{Q}_p with the Tate twist removed (cf. (3.2.8.2)). We also give a brief summary of the classification of CM formal groups, using the Scholze-Weinstein [25] classification as our starting point.

In Chapter 4, we explain how the quaternionic double coset

$$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p$$

arises as a moduli space of elliptic curves. Essentially, this section works out a very specific example of results of Caraiani-Scholze [4] on Igusa varieties.

In Chapter 5, we put together the ingredients introduced so-far to prove a version of Theorem B for classical modular forms.

In Chapter 6, we prove Theorem A using the results of Section 5 and a variant of Scholze's technique of fake Hasse invariants.

In Chapter 7, we explain an infinite level construction of overconvergent modular forms, then use this construction to extend the results of Section 5 to overconvergent modular forms and prove Theorem B.

Finally, in Chapter 8, we formulate a weak local-global compatibility conjecture with the local p -adic Jacquet-Langlands correspondences of Knight and Scholze, and discuss some consequences of this conjecture when combined with our other results.

1.6 Notation

1.6.1 Actions

For R a ring, the *standard* action of $M_n(R)$ on $\mathbb{A}^n(R) = R^n$ is by left multiplication of a column vector by a matrix, or, equivalently, by right multiplication of a row vector by the transposed matrix. The *dual* action of $M_n(R)$ on $\mathbb{A}^n(R) = R^n$ is by right multiplication of a row vector, or, equivalently, by left multiplication of a column vector by the transposed matrix. The standard action and dual action are interchanged by precomposition with matrix transpose.

For any group, we may interchange left and right actions by precomposing with an inverse.

When we say a group acts, we mean there is either a right or left action, which can be turned into the other by precomposition with an inverse.

If we have a set, a scheme, etc., equipped with an action of GL_n , then, by precomposition with matrix transpose, we obtain a *dual action*. This arises naturally in the following context: given a free R -module M , we may form the set $\mathrm{Isom}(R^n, M)$ of R -module isomorphisms between R^n and M (i.e., the set of trivializations, or bases of M). It admits standard and dual actions of $\mathrm{GL}_n(R)$ via precomposition. We may also form the set $\mathrm{Isom}(R^n, M^*)$, which is also equipped with standard and dual actions of $\mathrm{GL}_n(R)$ via precomposition. The dual basis gives a natural bijection

$$\mathrm{Isom}(R^n, M) \leftrightarrow \mathrm{Isom}(R^n, M^*)$$

and under this identification, the standard action on one set is identified with the dual action on the other.

In our work, this arises in the moduli interpretation of modular curves. It is equivalent to take, e.g., a trivialization of the singular homology of an elliptic curve or of the singular cohomology of an elliptic curve over \mathbb{C} , as these are canonically dual free \mathbb{Z} -modules. The standard action on the set of trivializations of the cohomology is identified with the dual action on the set of trivializations of the homology, and vice versa.

1.6.2 Adèles

We denote by \mathbb{A} the ring of adèles of \mathbb{Q} , by \mathbb{A}_f the ring of finite adèles, and, for p a prime, by $\mathbb{A}_f^{(p)}$ the ring of finite adèles away from p .

CHAPTER 2

GEOMETRIC PRELIMINARIES

In this section we discuss some geometric preliminaries. We focus on two main points: the realization of profinite sets as formal schemes / adic spaces, and the base change of perfectoid spaces from discretely valued fields to perfectoid fields. Combined, these two topics give rise to the theory of *twisted* profinite sets, which will play an important role later in Section 5.

That profinite sets live naturally in the world of adic spaces is well-known and straightforward; we give a short self-contained exposition. The second topic, on base change from discretely valued fields to perfectoid fields, is more subtle, and involves some of the intricacies of fiber product in the world of adic spaces – luckily, it has been studied by Kedlaya and Liu [18], whose results suffice for our purposes. We highlight here that the base change we consider involves an extra step of uniform completion. It is a fiber product, e.g., in the category of diamonds.

2.1 Perfectoid spaces

Here we fix our conventions for perfectoid spaces. We will only need to work in characteristic 0, but we will not want to fix a perfectoid base field, as at a certain point we will need to consider Galois actions coming from base extension from a discretely valued field to a perfectoid field. Thus, the most natural reference is Kedlaya and Liu [18, Sections 3.6 and 8.3].

As is standard in the subject, we use the language of adic spaces. For an introduction, we refer the reader to [28]. In this work, an adic space is always sheafy, i.e. an honest adic space in the language of [25].

Recall that an f -adic ring A is uniform if the ring of power-bounded elements A° is bounded.

Definition 2.1.1. [12] A *perfectoid algebra* is a uniform f -adic ring A containing a topolog-

ically nilpotent unit ϖ such that ϖ^p divides p in A° , and

$$\bar{\varphi} : A/\varpi \rightarrow A/\varpi^p$$

is surjective.

If A is a perfectoid algebra over \mathbb{Q}_p then A is stably uniform, and thus for A^+ a ring of integral elements, $\mathrm{Spa}(A, A^+)$ is an adic space (cf. [18, Theorem 3.6.5]). An adic space of this form is called *affinoid perfectoid*.

Definition 2.1.2. A *perfectoid space* is an adic space which can be covered by affinoid perfectoids.

2.1.3 Base change for perfectoid spaces

Given an adic space $X/\mathrm{Spa}(F, F^+)$ over an analytic field (F, F^+) , and an extension of analytic fields $(F, F^+) \rightarrow (F', F'^+)$, one would like to define a base change $X_{F'}$. The natural way to proceed is to take an affinoid (A, A^+) , form $A \widehat{\otimes}_F F'$ (and a suitable ring of integral elements inside), then glue. In general, however, it is not known that for a sheafy A , $A \widehat{\otimes}_F F'$ is also sheafy, so that the resulting base change may not be an (honest) adic space. It is known to be true, for example, if X is locally topologically of finite type over F or if X is perfectoid and F', F are both perfectoid (in which case $X_{F'}$ is also perfectoid).

The case of F discretely valued and F' perfectoid is notably absent, even for X perfectoid. In this case we encounter a perversity where for an affinoid perfectoid (A, A^+) over F , the completed tensor product $A \widehat{\otimes}_F F'$ may not be uniform (and thus, not perfectoid!) [18, Remark 2.8.5]. However, if we pass to the uniform completion (cf. [18, Definition 2.8.13]),

$$(A \widehat{\otimes}_F F')^u,$$

we obtain an affinoid perfectoid ([18, Corollary 3.6.18]). This can be glued to give a product

in the category of perfectoid spaces over F .

Definition 2.1.4. If F/\mathbb{Q}_p is discretely valued, $(F', F'^+)/ (F, \mathcal{O}_F)$ is a perfectoid field extension, and X/F is a perfectoid space, then we denote by $X_{F'}$ the product of X and $\mathrm{Spa}(F', F'^+)$ in the category of perfectoid spaces over F as above.

The uniform completion is functorial in continuous maps of F' , thus we obtain an action of $\mathrm{Aut}_{\mathrm{cont}}(F'/F)$ on $X_{F'}$.

Remark 2.1.5. In the language of diamonds, $X_{F'}^\diamond = X^\diamond \times_{\mathrm{Spd}F} \mathrm{Spd}F'$.

2.2 Profinite sets.

Let S be a profinite set. For any p -adically complete \mathbb{Z}_p -algebra R^+ , we define

$$S_{R^+} := \mathrm{Spf}\mathrm{Cont}(S, R^+).$$

Lemma 2.2.1. *For A a p -adically complete R^+ -algebra,*

$$S_{R^+}(A) = \mathrm{Cont}(\mathrm{Spf}A, S).$$

For (R, R^+) such that $\mathrm{Spa}(R, R^+)$ is a stably uniform adic space and R^+ is p -adically complete, we form the adic generic fiber

$$S_{(R, R^+)}^{\mathrm{ad}} := \mathrm{Spa}(\mathrm{Cont}(S, R), \mathrm{Cont}(S, R^+)).$$

The following result shows this construction is well-behaved:

Theorem 2.2.2. *If (R, R^+) is a stably uniform Huber pair, then so is*

$$(\mathrm{Cont}(S, R), \mathrm{Cont}(S, R^+)).$$

Proof. Evaluation at any point $s \in S$ induces a continuous section of $R \rightarrow \text{Cont}(S, R)$, and the result follows from [17, Lemma 1.2.18] \square

Remark 2.2.3. In the case that $(R, R^+) = (K, \mathcal{O}_K)$ for a non-archimedean field K , one can verify by hand that $S_{(K, \mathcal{O}_K)}$ is sheafy. In this case, the underlying topological space is homeomorphic to S , the structure sheaf \mathcal{O} evaluated on an open U is simply the ring of continuous functions on U , and the rational opens are the compact opens.

2.2.4 Twisted profinite sets

We will naturally encounter perfectoid spaces over a discretely valued field, which, after base change to a perfectoid field, become isomorphic to a profinite set. We will think of these as *twisted* profinite sets.

It will be helpful to keep in mind the following basic example:

Example 2.2.5. Let $\mathbb{Q}_p^{\text{cyc}} = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$. Then

$$\mathbb{Q}_p^{\text{cyc}} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^{\text{cyc}}$$

is not uniform. However, if we identify

$$\mathbb{Z}_p^\times = \text{Aut}_{\text{cont}}(\mathbb{Q}_p^{\text{cyc}}/\mathbb{Q}_p),$$

then

$$a \otimes b \rightarrow f : f(\sigma) = \sigma(a)b$$

extends to an isomorphism of uniform completions

$$(\mathbb{Q}_p^{\text{cyc}} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^{\text{cyc}})^u \xrightarrow{\sim} \text{Cont}(\mathbb{Z}_p^\times, \mathbb{Q}_p^{\text{cyc}}).$$

2.3 Equivariant geometry

We will consider spaces (schemes, complex analytic spaces, adic spaces, perfectoid spaces) equipped with actions of locally pro-finite groups.

2.3.1 Continuous actions

Following [22, Section 2], if X is a locally noetherian adic space or a perfectoid space, an action of a locally pro-finite group G on X is *continuous* if X admits a cover by affinoids (affinoid perfectoids) $\mathrm{Spa}(A, A^+)$ stabilized by compact opens $U \subset G$ such that the action map $U \times A \rightarrow A$ is continuous. By [22, Lemma 2.2], any quasi-compact open is then stabilized by some subgroup.

2.3.2 Equivariant sheaves

An equivariant sheaf on (the Zariski, analytic, étale, pro-étale site of) X is a sheaf \mathcal{F} equipped with isomorphisms $\cdot g^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ satisfying the obvious compatibilities.

Given a finite dimensional representation V of G on a F -vector space, we may form the constant sheaf \underline{V}^1 with natural G -action. If \mathcal{R} is a sheaf of rings over \underline{F} (e.g., \mathcal{O} , $\widehat{\mathcal{O}}$), and \mathcal{F} is an equivariant \mathcal{R} -module, an isomorphism of equivariant \mathcal{R} -modules

$$\underline{V} \otimes_K \mathcal{R} \xrightarrow{\sim} \mathcal{F}$$

is called an equivariant trivialization of \mathcal{F} . On equivariant \underline{F} -sheaves, \underline{V} represents

$$\mathcal{F} \mapsto \mathrm{Hom}_G(V, \mathcal{F}(X)).$$

1. On the pro-étale site, this should be formed with the discrete topology on V .

2.3.3 Towers

Let G be a locally profinite group. A G -tower is a projective system of spaces $(X_K)_K$ indexed by compact open subgroups $K \subset G$, and maps

$$\cdot g : X_K \rightarrow X_{g^{-1}Kg}$$

satisfying the natural compatibilities.

A vector bundle on $(X_K)_K$ is a compatible system of vector bundles $(\mathcal{V}_K)_K$ equipped with isomorphisms

$$\pi^* \mathcal{V}_K \xrightarrow{\sim} \mathcal{V}'_K$$

for $K' \subset K$ and $\pi : Y_{K'} \rightarrow Y_K$ the natural projection, compatible with compositions.

A G -equivariant vector bundle on $(X_K)_K$ is a vector bundle equipped with isomorphisms $\cdot g^* \mathcal{V} \rightarrow \mathcal{V}$ compatible with compositions.

2.3.4 Equivariant maps to \mathbb{P}^1

For a field F , we equip \mathbb{P}^1/F with the standard action of GL_2 . As a left action, it is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [x : y] = [ax + by : cx + dy].$$

The quotient map $\mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$ is equivariant for the standard action of GL_2 on $\mathbb{A}^2 \setminus \{0\}$. The sheaf of sections is the torsor of bases for $\mathcal{O}(-1)$, and is naturally equivariant. We equip $\mathcal{O}(k)$ with the induced equivariant structure for any k .

Example 2.3.5. For $k \geq 0$,

$$\Gamma(\mathcal{O}(k)) \cong \mathrm{Sym}^k(F^2)^*.$$

Let $\iota : H \rightarrow \mathrm{GL}_2$ be a map of groups, and equip \mathbb{P}^1 with the induced action of H . Let std denote the inflation of the standard representation of GL_2 to H via ι . Applying the

standard characterization of the functor of points of \mathbb{P}^1 , we find

Proposition 2.3.6. *Let X be an H -equivariant space. Giving an H -equivariant map $X \rightarrow \mathbb{P}^1$ is the same as giving an H -equivariant line bundle \mathcal{L} and an equivariant inclusion*

$$\mathcal{L} \hookrightarrow \text{std} \otimes \mathcal{O}$$

such that \mathcal{L} is locally a direct summand; dually, it is equivalent to giving an H -equivariant line bundle \mathcal{L}' and an equivariant surjection

$$\text{std}^* \otimes \mathcal{O} \rightarrow \mathcal{L}'.$$

2.4 Torsors, push-outs, and reduction of structure group

We will make use of the following construction: let X be an adic space over a non-archimedean field K/\mathbb{Q}_p and let G be a linear algebraic or affinoid group over K . Then G represents a sheaf of groups $G(\mathcal{O})$ on the étale or analytic site of X .

Example 2.4.1.

- For \mathbb{G}_m , the represented sheaf is \mathcal{O}^\times .
- For \mathbb{G}_m^1 , the annulus $|z| = 1 \subset \mathbb{G}_m$, the represented sheaf is $(\mathcal{O}^+)^\times$.

Let \mathcal{T} be a $G(\mathcal{O})$ -torsor on the étale or analytic site of X . Given an analytic representation

$$\rho : G \rightarrow GL(V)$$

we may form the push-out vector bundle

$$\mathcal{T} \times^\rho \mathcal{O} \otimes V,$$

which is the quotient of

$$\mathcal{T} \times \mathcal{O} \otimes V$$

by $(xg, v) \sim (x, \rho(g)v)$. Equivalently, one may define $\mathcal{T} \times^\rho \mathcal{O} \otimes V$ by taking a cover where \mathcal{T} is trivialized, then using the trivialization and ρ to define glueing data for a vector bundle on X .

One can also pass in the opposite direction – given a vector space V and a vector bundle \mathcal{V} locally isomorphic to $V \otimes \mathcal{O}$, we may form the *torsor of bases* $\mathcal{T}_{\mathcal{V}}$, which is the $GL(V \otimes \mathcal{O})$ -torsor defined by

$$\mathcal{T}_{\mathcal{V}}(U) = \text{Isom}((\mathcal{O} \otimes V)|_U, \mathcal{V}|_U).$$

Given an G -torsor \mathcal{T} , and a map $\rho : G \rightarrow H$, we may also form the push-out torsor

$$\mathcal{T} \times^\rho H(\mathcal{O}),$$

defined in a similar fashion. For example, if $\rho : H \rightarrow GL(V)$ is a representation, then

$$\mathcal{T} \times^\rho GL(V \otimes \mathcal{O})$$

is the torsor of bases for the vector bundle

$$\mathcal{T} \times^\rho V \otimes \mathcal{O}.$$

We will be particularly interested in the following setup: given a line bundle \mathcal{L} , we may form the \mathbb{G}_m -torsor of bases $\mathcal{T}_{\mathcal{L}}$. Pushing out $\mathcal{T}_{\mathcal{L}}$ by the irreducible representations of \mathbb{G}_m we recover the tensor powers of \mathcal{L} : the irreducible representations are the characters $\rho_k : z \mapsto z^k$, and

$$\mathcal{T}_{\mathcal{L}} \times^{\rho_k} \mathcal{O} \cong \mathcal{L}^k.$$

We will encounter situations where $\mathcal{T}_{\mathcal{L}}$ admits a *reduction of structure group* along a map of

map of analytic groups $r : G \rightarrow \mathbb{G}_m$. By a reduction of structure group along r , we mean the data of a G -torsor \mathcal{T} and an isomorphism

$$\mathcal{T} \times^r \mathbb{G}_m \xrightarrow{\sim} \mathcal{T}_{\mathcal{L}}.$$

Given such a reduction, we may form new bundles corresponding to representations $\rho : G \rightarrow GL(V)$ by taking the pushout

$$\mathcal{T} \times^{\rho} \mathcal{O} \otimes V.$$

We sometimes think of these new bundles as being generalized powers of our original bundle \mathcal{L} and write \mathcal{L}^{ρ} . Note however that reductions of structure group along a map r are not generally unique, so that \mathcal{L}^{ρ} depends on the choice of reduction.

Example 2.4.2.

- Giving a reduction of structure group of $\mathcal{T}_{\mathcal{L}}$ along

$$\mathbb{G}_m \xrightarrow{z \mapsto z^2} \mathbb{G}_m$$

is the same as giving a line bundle $\sqrt{\mathcal{L}}$ and an isomorphism $\sqrt{\mathcal{L}}^2 \xrightarrow{\sim} \mathcal{L}$. Given such a choice, we may form the half integral powers \mathcal{L}^k for $k \in \mathbb{Z}/2$.

- Giving a reduction of structure group of $\mathcal{T}_{\mathcal{L}}$ to \mathbb{G}_m^1 is the same as giving an integral structure on \mathcal{L} , i.e. a locally free of rank one sheaf of \mathcal{O}^+ modules, \mathcal{L}^+ , and an isomorphism $\mathcal{L}^+ \otimes_{\mathcal{O}^+} \mathcal{O} \xrightarrow{\sim} \mathcal{L}$. We do not obtain any new line bundles by taking characters of \mathbb{G}_m^1 , however, the choice of this reduction of structure group also equips each tensor power with an integral structure.

For $\epsilon < 1$ in the value group of K , we will consider the group $\mathbb{Z}_p^{\times, \epsilon}$, defined to be an ϵ -neighborhood of $\mathbb{Z}_p^{\times} \subset \mathbb{G}_m$. Concretely, this is given by taking coset representatives

$a_i \in (\mathbb{Z}_p/\epsilon)^\times$ and then taking the union of the affinoid balls of radius ϵ around each a_i (here \mathbb{Z}_p/ϵ is interpreted as \mathbb{Z}_p modulo the elements of \mathbb{Z}_p of absolute value $\leq \epsilon$). Given a reduction of structure group of \mathcal{L} along

$$\mathbb{Z}_p^{\times, \epsilon} \hookrightarrow \mathbb{G}_m$$

we may form \mathcal{L}^κ for any character κ of $\mathbb{Z}_p^{\times, \epsilon}$. We note that *any* continuous character of \mathbb{Z}_p^\times extends to $\mathbb{Z}_p^{\times, \epsilon}$ for some $\epsilon > 0$.

CHAPTER 3

THE GEOMETRY OF MODULAR CURVES

In this chapter we recall some aspects of the geometry of modular curves. We put a special emphasis on the construction of equivariant bundles on the tower of modular curves.

3.1 Modular curves and automorphic bundles

3.1.1 Modular curves

For a compact open subgroup $K \subset GL_2(\mathbb{A}_f)$ we denote by Y_K the modular curve of level K as a scheme over \mathbb{Q} and X_K its smooth compactification. For K sufficiently small, we give Y_K the following moduli interpretation on \mathbb{Q} -algebras R :

$$Y_K(R) = \{(E/R, \phi K)\} / \sim$$

where E/R is an elliptic curve up-to-isogeny and ϕK is a K -orbit of trivializations

$$\mathbb{A}_f^2 \rightarrow V_f E.$$

Here $V_f E$ is the rational adelic Tate module,

$$V_f E = \varprojlim_n E[n] \otimes \mathbb{Q}.$$

Remark 3.1.2. This isogeny moduli description can be interpreted literally using the pro-étale site of [3], or through the standard method of fixing a lattice in \mathbb{A}_f^2 preserved by K to define an equivalent moduli problem for elliptic curves up to isomorphism which makes use only of torsion sheaves and thus can be formulated on the étale site.

3.1.3 The tower of modular curves

We consider the tower $(Y_K)_K$ of modular curves as $K \subset GL_2(\mathbb{A}_f)$ varies over compact open subgroups. It admits a right action of $GL_2(\mathbb{A}_f)$ (cf. where $g \in GL_2(\mathbb{A}_f)$ acts by

$$\begin{aligned} \cdot g : Y_K(R) &\rightarrow Y_{g^{-1}Kg}(R) \\ (E/R, \phi K) &\mapsto (E/R, \phi Kg) = (E, (\phi g)g^{-1}Kg). \end{aligned}$$

This extends to an action on $(X_K)_K$.

3.1.4 Automorphic line bundles: moduli interpretation

Any object that can be constructed from an elliptic curve E up to isogeny give rises to an equivariant object over the tower (Y_K) . In particular, we will consider the line bundles formed naturally out of the Hodge cohomology of $\pi : E \rightarrow S = \text{Spec}R$. If we denote

$$\omega := R^0 \pi_* \Omega_{E/S}$$

the modular sheaf and

$$\det_{\text{dR}} := (R^1 \pi_* \Omega_{E/S})^*$$

then any such bundle is isomorphic as a $GL_2(\mathbb{A}_f)$ -equivariant bundle to

$$\omega^k \otimes \det_{\text{dR}}^t$$

for some k, t . These bundles have natural extensions to $(X_K)_K$ induce by the extension of ω by holomorphic q -expansions.

In particular, we note that by taking the second wedge power of the Hodge filtration on

deRham cohomology we obtain a canonical isomorphism

$$(R^1\pi_*\mathcal{O})^* = \omega \otimes \det_{\mathrm{dR}}. \quad (3.1.4.1)$$

3.1.5 Uniformization over \mathbb{C}

Over \mathbb{C} , we may consider the analytic covers $\tilde{Y}_{K,\mathbb{C}}$ classifying elliptic curves up to isogeny equipped with a trivialization

$$\psi : H_1(E, \mathbb{Q}) \rightarrow \mathbb{Q}^2$$

and a K -orbit of isomorphisms

$$\phi : \mathbb{A}_f^2 \rightarrow V_f E.$$

The trivialization ψ induces an isomorphism

$$H_1(E, \mathbb{A}_f) \xrightarrow{\sim} (\mathbb{A}_f)^2$$

and composing the with ϕ we obtain an element of $GL_2(\mathbb{A}_f)/K$. We may also pull back the Hodge filtration

$$\omega_{E^\vee} \hookrightarrow H_1(E, \mathbb{Q}) \otimes \mathbb{C}$$

via ψ to \mathbb{C}^2 to obtain a point in $\mathbb{P}^1(\mathbb{C})$. Combined, these maps induce an isomorphism

$$\tilde{Y}_{K,\mathbb{C}} \rightarrow (\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})) \times GL_2(\mathbb{A}_f)/K$$

The left action of $GL_2(\mathbb{Q})$ changing ψ via post-composition corresponds to the diagonal action on

$$(\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})) \times GL_2(\mathbb{A}_f)/K$$

and we find an analytic isomorphism

$$Y_{K,\mathbb{C}} \cong GL_2(\mathbb{Q}) \backslash (\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})) \times GL_2(\mathbb{A}_f) / K.$$

3.1.6 Automorphic bundles and uniformization

Let

$$P_{\text{std}} := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

We may view $\mathbb{P}^1(\mathbb{C}) = GL_2(\mathbb{C})/P^{\text{std}}$, and this choice identifies GL_2 -equivariant vector bundles on $\mathbb{P}^1(\mathbb{C})$ with representations of P^{std} . To go from a GL_2 -equivariant bundle to a P^{std} -representation, we take the fiber at P^{std} , which is fixed by P^{std} and thus admits an action. In the other direction, given a P^{std} vector bundle V we form

$$GL_2(\mathbb{C}) \times^{P^{\text{std}}} V := GL_2(\mathbb{C}) \times V / (gp, v) \sim (g, pv).$$

If V extends to a representation of G , then we obtain an isomorphism of equivariant bundles

$$\begin{aligned} \mathbb{P}^1 \times V &\xrightarrow{\sim} GL_2(\mathbb{C}) \times^{P^{\text{std}}} V & (3.1.6.1) \\ (gP^{\text{std}}, v) &\mapsto (g, g^{-1}v). \end{aligned}$$

Suppose given a pair of integers (s, t) . We may view it as the representation of P^{std} given by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a^s c^t$$

By restriction, we obtain a $GL_2(\mathbb{Q})$ -equivariant bundle $\mathcal{A}(s, t)$ on

$$(\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})) \times GL_2(\mathbb{A}_f)/K$$

which has a natural structure of as a $GL_2(\mathbb{Q})$, $GL_2(\mathbb{A}_f)$ -equivariant bundle on the tower

$$(\tilde{Y}_{K, \mathbb{C}})_K.$$

By the construction of the period map to $\mathbb{P}^1(\mathbb{C})$ via the Hodge filtration and the isomorphism $\Lambda^2 H^1(E, \mathbb{Q}) \otimes \mathbb{C} \cong H^2(E, \mathcal{O})$, we find that

$$\mathcal{A}(s, t) = \omega^{s-t} \det_{\mathrm{dR}}^s$$

as a $GL_2(\mathbb{Q})$, $GL_2(\mathbb{A}_f)$ -equivariant bundle. Quotienting by $GL_2(\mathbb{Q})$, we recover the $GL_2(\mathbb{A}_f)$ -equivariant bundles on $(Y_{K, \mathbb{C}})_K$ of 3.1.4.

3.2 Perfectoid modular curves

For $K^p \subset GL_2(\mathbb{A}_f^{(p)})$ a compact open, we denote by

$$Y_{\infty K^p}/\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$$

and

$$X_{\infty K^p}/\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$$

the infinite level perfectoid modular curves of tame level K^p as in [24, 4]. For (C, C^+) a complete algebraically closed extension of \mathbb{Q}_p ,

$$Y_{\infty K^p}(C, C^+) = \{E, \phi K^p\}$$

where E/C is an elliptic curve up-to-isogeny, and ϕK^p is a K^p -orbit of trivializations

$$\phi : \mathbb{A}_f^2 \rightarrow V_f E.$$

The towers

$$(Y_{\infty K^p})_{K^p} \text{ and } (X_{\infty K^p})_{K^p}$$

are equipped with $GL_2(\mathbb{A}_f) = GL_2(\mathbb{Q}_p) \times GL_2(\mathbb{A}_f^{(p)})$ -actions. The $GL_2(\mathbb{Q}_p)$ action preserves each individual level $Y_{\infty K^p}$ or $X_{\infty K^p}$ – these spaces are obtained by “going up the tower” at p . On a (C, C^+) point of $Y_{\infty K^p}$ as above, $GL_2(\mathbb{Q}_p)$ acts by changing the trivialization ϕ of $V_f E$ at p .

3.2.1 An equivariant trivialization of $\mathbb{Z}_p(-1)$

Let

$$\begin{aligned} \det_{\text{ur}} : GL_2(\mathbb{A}_f) &\rightarrow \mathbb{Q}^\times & (3.2.1.1) \\ \prod_l g_l &\mapsto \prod_l |\det g_l|_l \end{aligned}$$

Note that any compact subgroup $K \subset GL_2(\mathbb{A}_f)$ is contained in the kernel of \det_{ur} .

We also consider the determinant at p ,

$$\begin{aligned} \det_p : GL_2(\mathbb{A}_f) &\rightarrow \mathbb{Q}_p^\times & (3.2.1.2) \\ \prod_l g_l &\mapsto \det(g_p) \end{aligned}$$

Note that $\det_p \cdot \det_{\text{ur}}$, which a priori takes values in \mathbb{Q}_p^\times , in fact takes values in \mathbb{Z}_p^\times .

Theorem 3.2.2. *On $(Y_{\infty K^p})_{K^p}$, there is a canonical $GL_2(\mathbb{A}_f)$ -equivariant isomorphism*

$$\det_p \cdot \det_{\text{ur}} \cong \mathbb{Z}_p(1).$$

Proof. Over $Y_{\infty K^p}$ we have a universal elliptic curve up-to-isogeny E° equipped with a K^p -orbit of trivializations

$$\phi : \mathbb{A}_f^2 \rightarrow V_f E^\circ.$$

Fix a lattice $L \subset \mathbb{A}_f^2$ preserved by K^p . The lattice $\phi(L) \subset V_f E^\circ$ determines an elliptic curve E in the isogeny class E° , and via the fundamental class of E , an isomorphism

$$\Lambda^2(V_f E^\circ) \cong \mathbb{Q}_p(1).$$

Using this isomorphism, $\phi(e_1 \wedge e_2)$ gives a basis b_L of $\mathbb{Q}_p(1)$. If we take a second lattice L' , we find

$$b_{L'}/b_L = [\Lambda^2 L' : \Lambda^2 L].$$

Thus, $b_L/[\Lambda^2 L : \Lambda^2 \mathbb{A}_f^2]$ is a trivialization of $\mathbb{Q}_p(1)$ that does not depend on the choice of L .

We now verify the action: for $g \in GL_2(\mathbb{A}_f)$, $g^{-1}L$ is preserved by $g^{-1}K^p g$, and we find that for $(E, \phi g(g^{-1}K^p g))$,

$$b_{g^{-1}L} = (\det g_p) b_L.$$

Because

$$[\Lambda^2 g^{-1}L : \Lambda^2 \mathbb{Z}_p] = [\Lambda^2 g^{-1}L : \Lambda^2 L][\Lambda^2 L : \Lambda^2 \mathbb{Z}_p] = \det_{\text{ur}} g^{-1} [\Lambda^2 L : \Lambda^2 \mathbb{Z}_p].$$

we have,

$$\frac{b_{g^{-1}L}/[\Lambda^2 g^{-1}L : \Lambda^2 \mathbb{Z}_p]}{b_L/[\Lambda^2 L : \Lambda^2 \mathbb{Z}_p]} = \det g_p \cdot \det_{\text{ur}} g.$$

Thus we obtain a basis for $\mathbb{Q}_p(1)$ that transforms as desired under $GL_2(\mathbb{A}_f)$. Because $\det_p \cdot \det_{\text{ur}}$ takes values in \mathbb{Z}_p^\times , to see that this is in fact a basis for $\mathbb{Z}_p(1)$, it suffices to verify this for a single K^p in each conjugacy class of compact open. Taking $K^p \subset GL_2(\widehat{\mathbb{Z}})$ and $L = \widehat{\mathbb{Z}}^2$ this is clear. \square

Remark 3.2.3. This trivialization can also be deduced directly from the reciprocity law for

the connected components of the canonical model of the Shimura variety.

Remark 3.2.4. This construction has a natural archimedean analog: we may form the cover $Y_{\infty K, \mathbb{C}}$ of the complex analytic modular curve $Y_{K, \mathbb{C}}$ trivializing real singular cohomology. Over this space, there is a natural trivialization of $\mathbb{R}(1)$. It can be constructed via the moduli interpretation in the p-adic case, or, alternatively, as follows: $Y_{\infty K, \mathbb{C}}$ has a natural uniformization

$$GL_2(\mathbb{Q}) \backslash X \times GL_2(\mathbb{R}) \times GL_2(\mathbb{A}_f) / K^P$$

where $GL_2(\mathbb{R})$ is equipped with the discrete topology. Over X we have a natural trivialization of $\mathbb{Z}(1)$, but it is not $GL_2(\mathbb{Q})$ -invariant (it transforms via the sign of the determinant). If we extend this trivialization to $\mathbb{R}(1)$, then multiply it by $\det_{\mathbb{R}} \cdot \det_{\text{ur}}$, it becomes $GL_2(\mathbb{Q})$ -invariant, as desired. It transforms under the $GL_2(\mathbb{A}_f)$ -action as $\det_{\mathbb{R}} \cdot \det_{\text{ur}}$.

Of course, in the archimedean case we can simplify this greatly – instead of involving the finite places, we could simply multiply by $\text{sgn}(\det_{\mathbb{R}})$ to obtain a trivialization of $\mathbb{Z}(1)$ invariant under $GL_2(\mathbb{Q})$, transforming as \det under $GL_2(\mathbb{R})$, and trivially under $GL_2(\mathbb{A}_f)$. This is possible because there is a unique basis element for $\mathbb{Z}(1)$ in each connected component of $\mathbb{R}(1)$.

3.2.5 Determinant bundles

We define two $GL_2(\mathbb{A}_f)$ -equivariant bundles on $(Y_{\infty K^P})_{K^P}$:

$$\begin{aligned} \det_{\text{dR}} &:= (R^1 \pi_* \Omega_{E/Y_{\infty K^P}})^{-1} \text{ and} \\ \det_{\text{HT}} &:= (R^2 \pi_* \widehat{\mathcal{O}})^{-1} \end{aligned}$$

We have a canonical Hodge-Tate comparison isomorphism

$$\det_{\text{dR}}(1) = \det_{\text{HT}}. \tag{3.2.5.1}$$

Lemma 3.2.6. *There are canonical $GL_2(\mathbb{A}_f)$ -equivariant isomorphisms*

$$\det_p \otimes \widehat{\mathcal{O}} \cong \det_{\text{HT}}, \quad (3.2.6.1)$$

and

$$\det_{\text{ur}}^{-1} \otimes \mathcal{O} \cong \det_{\text{dR}}. \quad (3.2.6.2)$$

Proof. The isomorphism (3.2.6.1) follows from

$$\Lambda^2 V_p E \otimes \widehat{\mathcal{O}} \cong \det_{\text{HT}}$$

by taking the basis $\phi_p(e_1 \wedge e_2)$. The isomorphism (3.2.6.2) then follows by combining (3.2.6.1), (3.2.5.1), and Theorem 3.2.2. \square

3.2.7 The Hodge-Tate period map

By [24], there are $GL_2(\mathbb{Q}_p)$ -equivariant Hodge-Tate period maps

$$\pi_{\text{HT}} : X_{\infty K^p} \rightarrow \mathbb{P}^1$$

which fit into a $GL_2(\mathbb{A}_f)$ -equivariant map of towers

$$(X_{\infty K^p})_{K_p} \rightarrow (\mathbb{P}^1)_{K_p}$$

where the right-hand side is equipped with the trivial $GL_2(\mathbb{A}_f^{(p)})$ action (i.e. the maps $\cdot g : \mathbb{P}^1_{K^p} \rightarrow \mathbb{P}^1_{g^{-1}K^p g}$ are the identity) and the standard $GL_2(\mathbb{Q}_p)$ -action at each level.

Remark 3.2.8. Recall that, by our conventions, we switch freely between left and right actions by precomposing with an inverse.

The map π_{HT} can be interpreted as follows¹: over $Y_{\infty K^p}$, we have the universal elliptic curve up-to-isogeny $\pi : E^\circ \rightarrow Y_{\infty K^p}$. Let std_p be the representation of $\text{GL}_2(\mathbb{A}_f)$ inflated from the standard representation of $\text{GL}_2(\mathbb{Q}_p)$ on \mathbb{Q}_p^2 . We obtain a canonical equivariant trivialization

$$\text{std}_p \otimes \widehat{\mathcal{O}}^2 \rightarrow V_p E^\circ \otimes \widehat{\mathcal{O}} \quad (3.2.8.1)$$

coming from the canonical canonical trivialization

$$\mathbb{Q}_p^2 \rightarrow V_p E^\circ.$$

There is a canonical Hodge-Tate filtration

$$0 \rightarrow \omega^{-1}(1) \rightarrow V_p E^\circ \otimes \widehat{\mathcal{O}} \rightarrow (R^1 \pi_* \mathcal{O} \otimes \widehat{\mathcal{O}})^* \rightarrow 0$$

Using the trivialization (3.2.8.1) on the middle term, the trivialization of the Tate twist from Theorem 3.2.2 on the first term, (3.1.4.1) and (3.2.6.2) on the last term, and then restricting to the analytic site, we obtain

$$0 \rightarrow \det_{\text{ur}} \cdot \det_p \otimes \omega^{-1} \rightarrow \text{std}_p \otimes \mathcal{O} \rightarrow \det_{\text{ur}}^{-1} \otimes \omega \rightarrow 0. \quad (3.2.8.2)$$

The map π_{HT} is then induced (cf. Proposition 2.3.6) by the inclusion

$$\det_{\text{ur}} \cdot \det_p \otimes \omega^{-1} \hookrightarrow \text{std}_p \otimes \mathcal{O}.$$

Because π_{HT} is locally constant in a neighborhood of the boundary (in fact, it is locally constant on the entire ordinary locus), it extends naturally to $(X_{K^p})_{K^p}$.

The following theorem is an immediate consequence of the construction of π_{HT} via

1. One should be slightly careful here, as one step in showing the existence of $Y_{\infty K^p}$ as a perfectoid space is constructing π_{HT} . However, accepting this existence, there is no problem in interpreting the maps as we do.

(3.2.8.2). It is a p -adic analog of the archimedean construction of automorphic bundles at infinite level in 3.1.6

Theorem 3.2.9. *As $GL_2(\mathbb{A}_f)$ -equivariant bundles on $(X_{\infty K^p})_{K^p}$,*

$$\omega^k \det_{\mathrm{dR}}^m = \pi_{\mathrm{HT}}^* \left(\det_p^k \cdot \det_{\mathrm{ur}}^{k-m} \otimes \mathcal{O}(k) \right).$$

3.2.10 Affinoid perfectoids at infinite level

For s_1, s_2 a basis of $H^0(\mathbb{P}_{\mathbb{Q}_p}^1, \mathcal{O}(1))$, define an affinoid subset $U_{s_1, s_2} \subset \mathbb{P}_{\mathbb{Q}_p}^1$ by $|s_1| \leq |s_2|$. By the results of [24], we find

Theorem 3.2.11. *Let W be a rational sub-domain of $U = U_{s_1, s_2}$ for some s_1, s_2 as above. Then $\pi_{\mathrm{HT}}^{-1}(W)$ is affinoid perfectoid, and for $K_p \subset GL_2(\mathbb{Q}_p)$ sufficiently small,*

$$\pi_{\mathrm{HT}}^{-1}(W)$$

is the preimage of an affinoid

$$W_{K_p} \subset X_{K_p K^p},$$

and

$$\varinjlim_{K_p} H^0(W_{K_p}, \mathcal{O})$$

is dense in

$$H^0(\pi_{\mathrm{HT}}^{-1}(W), \mathcal{O}).$$

Proof. Using the $GL_2(\mathbb{Q}_p)$ -equivariance of π_{HT} , we may assume $s_1 = X$ and $s_2 = Y$, the standard basis. If $W = U$ we then conclude by [24, Theorem III.3.17-(i)]. For a general rational sub-domain $W \subset U$, $\pi_{\mathrm{HT}}^{-1}(W)$ is affinoid perfectoid as a rational sub-domain of the affinoid perfectoid $\pi_{\mathrm{HT}}^{-1}(U)$. Moreover, by the density statement for functions at finite level on U , we can choose functions coming from finite level to define the rational sub-domain,

and thus W is also the preimage of affinoids at sufficiently small finite level. The density of functions at finite level for W then follows from the density for U and the definition of the ring of functions on a rational subdomain. \square

3.3 Hecke operators

Given an equivariant vector bundle \mathcal{V} on the tower (X_K) in the standard way we obtain, for any fixed K , an action of the abstract double coset Hecke algebra

$$\mathbb{Z}[\mathrm{GL}_2(\mathbb{A}_f)//K]$$

on $H^0(Y_K, \mathcal{V})$ (or $H^0(X_K, \mathcal{V})$).

Remark 3.3.1. The standard Hecke action on weight k modular forms comes from the equivariant bundle

$$\omega^k \otimes \det_{\mathrm{dR}} = \omega^{k-2} \otimes \Omega_X(\log \text{cusps}).$$

The reason is that for $k \geq 2$, this matches the action induced by the inclusion

$$H^0(X, \omega^{k-2} \otimes \Omega_X(\log \text{cusps})) \hookrightarrow H^1(Y_K, \mathrm{Sym}^{k-2}(\mathbb{C}^2)^*).$$

The same applies if we replace $(X_K)_K$ with any tower of opens $U_K \subset (X_K)_K$ such that the transition maps are finite étale, and similarly for the perfectoid tower $(X_{\infty K^p})_{K^p}$ and the prime-to- p Hecke algebra. For a discussion of the trace maps used in the setting of adic spaces, we refer to the beginning of [24, IV.3].

3.4 Lubin-Tate space and CM formal groups

We summarize some well-known facts about height two CM formal groups using the Scholze-Weinstein classification [25] as our starting point.

3.4.1 The Scholze-Weinstein classification

For a p -divisible group G over $\mathcal{O}_{\mathbb{C}_p}$, there is a Hodge-Tate exact sequence

$$0 \rightarrow \mathrm{Lie}G(1)[1/p] \rightarrow T_pG \otimes \mathbb{C}_p \rightarrow \omega_{G^\vee}[1/p] \rightarrow 0. \quad (3.4.1.1)$$

The sequence (3.4.1.1) is functorial in G .

By a theorem of Scholze-Weinstein [25, Theorem B]², the assignment

$$G \mapsto (\mathrm{Lie}G(1)[1/p], T_pG \otimes \mathbb{C}_p)$$

induces an equivalence of categories between the category of p -divisible groups over $\mathcal{O}_{\mathbb{C}_p}$ and the category of pairs

$$(W, T)$$

where T is a free \mathbb{Z}_p -module and $W \subset T \otimes \mathbb{C}_p$ is a subspace. The dimension of G corresponds to the dimension of G , and the height of G to the rank of T .

3.4.2 The classifying point

Let G be a one-dimensional height 2 p -divisible group over $\mathcal{O}_{\mathbb{C}_p}$, and let

$$\mathrm{End}^\circ(G) = \mathrm{End}(G) \otimes \mathbb{Q}_p.$$

We fix a trivialization

$$\mathrm{triv} : \mathbb{Z}_p^2 \rightarrow T_pG,$$

from which we obtain a map $j_M : \mathrm{End}^\circ(G) \rightarrow M_2(\mathbb{Q}_p)$ such that

$$\mathrm{triv}_{\mathbb{Q}_p} \circ j_M(a) = a_* \circ \mathrm{triv}_{\mathbb{Q}_p}.$$

2. In [25], the theorem is stated with a Tate twist on T_pG instead of $\mathrm{Lie}G[1/p]$

In the notation of 2.3.4 (using the map j_M to GL_2), triv induces an $\mathrm{End}^\circ(G)^\times$ -equivariant trivialization over the point $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$

$$\mathrm{std} \otimes \mathcal{O} \xrightarrow{\sim} T_p G \otimes \mathcal{O}.$$

Because the Hodge-Tate filtration $\mathrm{Lie}G[1/p](1) \subset T_p G \otimes \mathbb{C}_p$ is preserved by a_* , Proposition 2.3.6 gives a $\mathrm{End}^\circ(G)^\times$ -equivariant map from $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ to \mathbb{P}^1 . Equivalently, we obtain a point $x \in \mathbb{P}^1(\mathbb{C}_p)$ stabilized by the action of $\mathrm{End}^\circ(G)^\times$ through j_M and a natural identification

$$\begin{array}{ccc} (T_p G)^* \otimes \mathbb{C}_p & \longrightarrow & \omega_G[1/p](-1) \\ \downarrow \sim & & \downarrow \sim \\ \Gamma(\mathcal{O}(1)) & \longrightarrow & \Gamma(\mathcal{O}(1)|_x) \end{array}$$

identifying the left action of $j_M(a)$ on the bottom row with the map $(a^{-1})^*$ on the top row.

We refer the the point x as the *classifying point* of the pair (G, triv) .

3.4.3 CM formal groups

Definition 3.4.4. Let F/\mathbb{Q}_p be a quadratic extension. For L/F a complete extension, we say a height 2 p -divisible group G over \mathcal{O}_L has complex multiplication (CM) by F if there is an embedding $F \rightarrow \mathrm{End}^0(G)$ such that F acts by the identity character on $\omega_G[1/p]$ (equivalently, it acts by the identity character on $\mathrm{Lie}G[1/p]$). The embedding is unique if it exists, in which case $F \cap \mathrm{End}(G)$ is an order in F , which we call the CM order.

Remark 3.4.5. If G has CM by F , then we will speak of the action of F on G by quasi-isogenies, in which case we are always referring to the normalization as above where the pull-back action of F on $\omega_G[1/p]$ is via the identity character.

Using the construction of 3.4.2, we find that G/\mathbb{C}_p has CM by F if and only if for some

(equivalently, any) choice of triv , the classifying point x is contained in

$$\Omega(F) := \mathbb{P}^1(F) \setminus \mathbb{P}^1(\mathbb{Q}_p).$$

Lemma 3.4.6. *The Hodge-Tate sequence induces a bijection between isomorphism classes of height 2 p -divisible groups over $\mathcal{O}_{\mathbb{C}_p}$ with CM by F and $\text{GL}_2(\mathbb{Z}_p)$ orbits on $\Omega(F)$.*

From this we deduce

Corollary 3.4.7. *There is a unique isogeny class of height 2 p -divisible groups over $\mathcal{O}_{\mathbb{C}_p}$ with CM by F . For each order $R \subset F$, there is a unique isomorphism class of height 2 p -divisible group over $\mathcal{O}_{\mathbb{C}_p}$ with CM order R .*

3.4.8 Lubin-Tate groups and the Galois action

For $[F : \mathbb{Q}_p] = 2$, the theory of Lubin-Tate formal groups shows there is a unique height 2 p -divisible group G over $\mathcal{O}_{\check{F}}$ with CM by F . The CM order is \mathcal{O}_F , and the assignment which sends an element

$$\sigma \in \text{Aut}_{\text{cont}}(\widehat{F^{\text{ab}}}/\check{F}) = \text{Gal}(F^{\text{ab}} \cdot \check{F}/\check{F})$$

to the endomorphism a_σ such that σ acts as a_{σ^*} on $T_p G$ is an isomorphism onto \mathcal{O}_F^\times . We note that, for the standard left Galois action on $T_p G^*$, σ acts as $(a_\sigma^{-1})^*$.

If $R \subset \mathcal{O}_F$ is an order, there is a corresponding F^\times -orbit of sub-lattices inside $T_p G[1/p]$ whose stabilizer in \mathcal{O}_F is R . Let

$$\check{F}_{R^\times} := \widehat{F^{\text{ab}}}^{R^\times},$$

the fixed field of $R^\times \subset \mathcal{O}_K^\times$ acting via Galois on $\widehat{F^{\text{ab}}}$. Then there is a unique G' defined over $\mathcal{O}_{\check{F}_{R^\times}}$ which is isogenous to G and has CM order R . The Galois action of R^\times on $T_p G'$ is via endomorphisms as before.

3.4.9 Lubin-Tate deformation space

Let G_0 be the unique height 2 formal group over $\overline{\mathbb{F}_p}$, which has quasi-isogenies by $D^\times(\mathbb{Q}_p)$.

There is a Lubin-Tate formal scheme

$$\text{LT}/\text{Spf}\check{\mathbb{Z}}_p$$

such that for $R \in \text{Comp}_{\check{\mathbb{Z}}_p}$,

$$\text{LT}(R) = \{(G, \rho)\} / \sim$$

where G/R is a formal group and $\rho : G_{0,R/p} \rightarrow G_{R/p}$ is a quasi-isogeny, all considered up to isomorphism of G . It admits a right action of $D^\times(\mathbb{Q}_p)$ by pre-composition.

If we take a formal group G with CM order R over $\mathcal{O}_{\check{F}_{R^\times}}$ as in 3.4.8 then, modulo a uniformizer π , it is isomorphic to G_0 . Any such isomorphism lifts uniquely to a quasi-isogeny modulo p , and thus determines a point of $m \in \text{LT}(\mathcal{O}_{\check{F}_{R^\times}})$. The choice of an isomorphism ρ_0 also induces a map

$$j_D : R \rightarrow D(\mathbb{Q}_p)$$

such that

$$\rho_0 \circ j_D(r) = r \circ \rho_0.$$

By taking the orbit of m under $D^\times(\mathbb{Q}_p)$, we may identify the locus of LT where the deformation G has CM order R with the locally profinite set $j_D(R^\times) \backslash D^\times(\mathbb{Q}_p)$.

3.4.10 Lubin-Tate space at infinite level

As in [25, Definition 6.3.3], we consider the perfectoid Lubin-Tate space $\widehat{\text{LT}}_\infty / \text{Spa}(\check{\mathbb{Q}}_p, \check{\mathbb{Z}}_p)$. Its functor of points sends an affinoid perfectoid $(\check{\mathbb{Q}}_p, \check{\mathbb{Z}}_p)$ -algebra (R, R^+) to triples

$$(G, \rho, \alpha)$$

where $(G, \rho) \in \text{LT}(R^+)$, and

$$\alpha : \mathbb{Z}_p^2 \xrightarrow{\sim} T_p G$$

(here $T_p G$ is interpreted as a sheaf on the pro-étale site of $\widehat{\text{LT}}_\infty$.)

Remark 3.4.11. In [25, 4] a version of $\widehat{\text{LT}}_\infty$ before completion is considered, but we will not need this.

There is a Hodge-Tate period map

$$\pi_{\text{HT}} : \widehat{\text{LT}}_\infty \rightarrow \mathbb{P}^1$$

measuring the position of the Hodge-Tate filtration with respect to this trivialization. On \mathbb{C}_p -points, it admits the following description: a \mathbb{C}_p -point of $\widehat{\text{LT}}_\infty$ corresponds to a triple (G, ρ, triv) where $G/\mathcal{O}_{\mathbb{C}_p}$ is a p -divisible group,

$$\rho : G_{0, \mathcal{O}_{\mathbb{C}_p}/p} \rightarrow G_{\mathcal{O}_{\mathbb{C}_p}/p}$$

is a quasi-isogeny, and $\text{triv} : \mathbb{Z}_p^2 \rightarrow T_p G$ is an isomorphism. Then π_{HT} sends this point to the classifying point of (G, triv) as in 3.4.2.

CHAPTER 4

$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_F) / K^P$ AS A GEOMETRIC OBJECT

As in 2.2, we may view the profinite set $D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_F) / K^P$ as a formal scheme over $\mathrm{Spf} \check{\mathbb{Z}}_p$. In this section we explain how it can be identified with a moduli space of supersingular curves with extra structure (an Igusa variety as in [4]). Using this moduli interpretation and Serre-Tate theory, we explain how fibers of the Hodge-Tate period map are naturally twisted versions of $D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_F) / K^P$.

Except for the identification of $D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_F) / K^P$ with the Igusa variety, this amounts to a very special case of results of Caraiani-Scholze [4, Section 4.3]. We repeat some of their arguments because it will be useful to have some maps written down explicitly for later use. The identification of $D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_F) / K^P$ with the Igusa variety is likely also well-known to experts, though maybe not in this precise form.

4.1 Igusa varieties

4.1.1 Serre-Tate theory

We recall a formulation of Serre-Tate deformation theory for elliptic curves, as explained, e.g., in the first section of [15].

For R a p -adically complete \mathbb{Z}_p algebra, we consider the groupoid Ell_R° whose objects are elliptic curves over R and whose morphisms are quasi-isogenies.

We also consider the groupoid Def_R° of triples (H, E_0, ρ) where H/R is a p -divisible group, $E_0/(R/p)$ is an elliptic curve,

$$\rho : H_{R/p} \rightarrow E_0[p^\infty]$$

is a quasi-isogeny, and morphisms are given by quasi-isogenies in H and quasi-isogenies of E_0 intertwining ρ .

Theorem 4.1.2. *The functor*

$$E \mapsto (E[p^\infty], E_{R/p}, \text{Id})$$

from Ell_R° to Def_R° is an equivalence.

Proof. An inverse functor is given by taking a triple (H, E_0, ρ) , replacing E_0 with the unique p -power isogenous elliptic curve E' over R/p such that ρ factors as

$$H_{R/p} \xrightarrow{\rho'} E'[p^\infty] \rightarrow E[p^\infty]$$

for ρ' an isomorphism, then taking the Serre-Tate lift of E' determined by ρ' as in [15, Theorem 1.2.1]. □

4.1.3 Igusa schemes

Let \mathbb{X} be a height 2 p -divisible group over $\overline{\mathbb{F}}_p$. The (big) Igusa scheme $\text{Ig}_{\mathbb{X}, K^p}$, as introduced in [4, Section 4.3] in a more general PEL setting, is the affine perfect scheme over $\overline{\mathbb{F}}_p$ whose points in an $\overline{\mathbb{F}}_p$ -algebra R classify elliptic curves E/R equipped with level K^p structure and a quasi-isogeny $\mathbb{X}_R \rightarrow E[p^\infty]$, considered up to quasi-isogeny of E .

4.1.4 Igusa formal schemes

The (big) Igusa formal scheme $\text{Ig}_{\mathbb{X}, K^p, \check{\mathbb{Z}}_p}$ over $\text{Spf} \check{\mathbb{Z}}_p$ is Spf of the Witt vectors of the perfect ring underlying $\text{Ig}_{\mathbb{X}, K^p}$. By the universal property of Witt vectors, for $R' \in \text{Nilp}_{\check{\mathbb{Z}}_p}$,

$$\text{Ig}_{\mathbb{X}, K^p, \check{\mathbb{Z}}_p}(R) = \{(E, \rho, \alpha)\} / \sim \tag{4.1.4.1}$$

where $E/(R/p)$ is an elliptic curve,

$$\rho : \mathbb{X}_{R/p} \rightarrow E[p^\infty]$$

is a quasi-isogeny, and α is a level K^p structure on E , all considered up to quasi-isogeny of E . It admits a right action of the group of quasi-isogenies of \mathbb{X} by precomposition (if \mathbb{X} is ordinary then it admits an action of a larger sheaf of groups, however, this will play no role for us).

4.1.5 A second moduli description

If R is a p -adically complete $\check{\mathbb{Z}}_p$ algebra and G is a p -divisible group over R equipped with a quasi-isogeny

$$\rho_0 : G \times_R R/p \rightarrow \mathbb{X} \times_{\overline{\mathbb{F}}_p} R/p$$

then Serre-Tate theory (as in Theorem 4.1.2) produces an alternative moduli description for

$$\mathrm{Ig}_{\mathbb{X}, K^p, R} := \mathrm{Ig}_{\mathbb{X}, K^p, \check{\mathbb{Z}}_p} \times_{\check{\mathbb{Z}}_p} \mathrm{Spf} R$$

(cf. [4, Lemma 4.3.10]): for $R' \in \mathrm{Comp}_R$,

$$\mathrm{Ig}_{\mathbb{X}, K^p, R}(R') = \{(E, \rho, \alpha)\} / \sim \tag{4.1.5.1}$$

where E/R' is an elliptic curve,

$$\rho : E[p^\infty] \rightarrow G \times_R R'$$

is a quasi-isogeny, and α is a level K_p structure, all considered up to quasi-isogeny of E . To pass from the mod p description (4.1.4.1) to the isomorphism description (4.1.5.1), we replace E over (R/p) with the Serre-Tate lift determined by $\rho \circ \rho_{0, R'/p}$ by Theorem 4.1.2.

We note that ρ_0 determines a map

$$j : \mathrm{End}^\circ(G) \rightarrow \mathrm{End}^\circ(\mathbb{X} \times_{\overline{\mathbb{F}}_p} R/p)$$

such that

$$\rho_0 \circ a = j(a) \circ \rho_0.$$

In particular, the natural action of the quasi-isogenies of G on the second moduli interpretation (4.1.5.1) is identified with the action through composition with j on the original moduli problem.

4.2 The supersingular Igusa formal scheme

Recall that in 1.2.1 we fixed a super-singular elliptic curve E_0 over $\overline{\mathbb{F}_p}$ equipped with a trivialization of its prime-to- p Tate module

$$\mathbb{T}_{\widehat{\mathbb{Z}}^{(p)}} E_0 \cong (\widehat{\mathbb{Z}}^{(p)})^2.$$

The p -divisible group $E_0[p^\infty]$ is the unique up to isomorphism height 2 formal group over $\overline{\mathbb{F}_p}$ and we have a natural identification of $D^\times(\mathbb{Q}_p)$ with the quasi-isogenies of $E_0[p^\infty]$.

We show,

Theorem 4.2.1. *For each K^p , the data above determines a natural $D^\times(\mathbb{Q}_p)$ -equivariant isomorphism of formal schemes over $\mathrm{Spf}\check{\mathbb{Z}}_p$*

$$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p \xrightarrow{\sim} \mathrm{Ig}_{E_0[p^\infty], K^p, \check{\mathbb{Z}}_p}.$$

Furthermore, for varying K^p these fit into a $D^\times(\mathbb{A}_f)$ -equivariant isomorphism of towers of formal schemes over $\mathrm{Spf}\check{\mathbb{Z}}_p$

$$(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p)_{K^p} \xrightarrow{\sim} (\mathrm{Ig}_{E_0[p^\infty], K^p, \check{\mathbb{Z}}_p})_{K^p}.$$

Proof. To ease notation, we will denote the formal scheme

$$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p$$

by S_{K^p} . We also fix a choice of a continuous section of topological spaces

$$s : D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p \rightarrow D^\times(\mathbb{A}_f) / K^p \cong D^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^{(p)}) / K^p \quad (4.2.1.1)$$

Using this section, we define a map

$$S_{K^p} \rightarrow \mathrm{Ig}_{E_0[p^\infty], K^p, R}$$

as follows: for $R' \in \mathrm{Comp}R$, an element of $S_{K^p}(R')$ is a continuous map

$$g : \mathrm{Spf}R' \rightarrow S_{K^p}.$$

Composing with the section (4.2.1.1) and the projections onto the two components, we obtain maps

$$g_1 : \mathrm{Spf}R' \rightarrow D^\times(\mathbb{Q}_p)$$

and

$$g_2 : \mathrm{Spf}R' \rightarrow D^\times(\mathbb{A}_f^{(p)}) / K^p.$$

Since the latter set is discrete, $s \circ g = g_1 \times g_2$. Furthermore, g_1 gives a quasi-isogeny of $E_0[p^\infty]_{R'/p}$. Thus, we may define a point in $\mathrm{Ig}_{E_0[p^\infty]}(R')$ by $(E_{0,R'/p}, g_1, g_2)$.

Any two sections s differ by an element of $h \in D^\times(\mathbb{Q})$, and thus give rise to the same map, since

$$(E_{0,R'/p}, g_1, g_2) \sim (E_{0,R'/p}, hg_1, hg_2)$$

Similarly we find the map is injective. To verify surjectivity, it will suffice to see that every

elliptic curve E over R/p with level K^p structure whose p -divisible group is quasi-isogenous to $E_0[p^\infty]$ is itself quasi-isogenous to $E_{0,R'/p}$.

For such a curve, the Hasse-invariant generates (locally) a nilpotent ideal I of R/p . Thus $E_{R/I}$ with its level K^p structure is classified by a map to a finite reduced subscheme of the modular curve of level K^p over $\overline{\mathbb{F}}_p$, the super-singular locus. In particular, $\text{Spec}R/I$ is a disjoint union of open subschemes where E is isomorphic to the base change of a supersingular curve over $\overline{\mathbb{F}}_p$. Any supersingular curve over $\overline{\mathbb{F}}_p$ is isogenous to E_0 , and any such isogeny mod I lifts to a quasi-isogeny over R/p , and thus we conclude.

The $D^\times(\mathbb{Q}_p)$ and $D^\times(\mathbb{A}_f^{(p)})$ equivariance are straightforward consequences of the construction. □

4.3 Fibers of the Hodge-Tate period map

Let (K, K^+) be a non-archimedean field over $\check{\mathbb{Q}}_p$, and let $x \in \Omega(K, K^+)$. We may form the fiber of π_{HT} over x inside the infinite level modular curve, X_{x, K^p} .

Suppose given a perfectoid extension (C, C^+) of (K, K^+) and a point $x_\infty \in \text{LT}_\infty(C, C^+)$ above x corresponding to (G, ρ_0, triv) . As in [4] we define a map

$$\text{Ig}_{E_0[p^\infty], K^p, (C, C^+)} \rightarrow X_{x, K^p, (C, C^+)} \tag{4.3.0.1}$$

as follows: using the second moduli description of 4.1.5, for a perfectoid algebra $(R, R^+)/(C, C^+)$, an element of

$$\text{Ig}_{E_0[p^\infty], K^p, (C, C^+)}(R, R^+)$$

is given by a triple (E, ρ, α) , where E/R^+ is an elliptic curve and

$$\rho : G_{R^+} \rightarrow E[p^\infty]$$

is a quasi-isogeny.

This is mapped to the element of $X_{\infty K^p, (C, C^+)}(R, R^+)$ given by the triple

$$(E, \rho_* \circ \text{triv}, \alpha).$$

To see that this factors through the fiber $X_{x, K^p, (C, C^+)}$ it suffices to check on geometric points, where it is clear.

Theorem 4.3.1. *The map (5.3.0.1) is an isomorphism of perfectoid spaces.*

Proof. This follows from [4, Lemma 4.3.20] (cf. also [4, Definition 4.3.17]), plus the fact that a perfectoid space is determined by its points in perfectoid spaces. \square

Remark 4.3.2. There is no need to restrict to perfectoid fields in the statement of Theorem 4.3.1. Taking the perfectoid Lubin-Tate space at infinite level, we obtain a natural formulation of the uniformization for the super-singular locus at infinite level.

CHAPTER 5

EVALUATING MODULAR FORMS AT CM POINTS

In this section we explain how to evaluate classical modular forms on the double coset $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K^p$. The main result of the section is Theorem 5.7.2, which says that the maps we construct are Hecke-equivariant and describes the action of a maximal torus in $D^\times(\mathbb{Q}_p)$ on their image in terms of the weight.

Our strategy is as follows: Using Theorem 4.2.1, we realize $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K^p$ as an Igusa formal scheme, which, after base-change to a perfectoid field, can be identified with a fiber of π_{HT} over a point in Ω via Theorem 4.3.1. Over this point in Ω , the sheaf $\mathcal{O}(1)$ has a natural trivialization, which pulls back via π_{HT} to a trivialisaton of ω . Using this trivialization we obtain functions on the double coset (with values in a very large extension of \mathbb{Q}_p).

The construction can be made on the fiber above any point $x \in \Omega(\mathbb{C}_p)$, however, there are considerable gains to be had by working over a point $x \in \Omega(F)$ for a quadratic extension F/\mathbb{Q}_p . These points correspond to CM formal groups, and using a reciprocity law we are able to obtain control over the action of a compact open inside a maximal torus $F^\times \subset D^\times(\mathbb{Q}_p)$ and over the field of coefficients.

Although we only treat classical modular forms in this section, the method will generalize easily to overconvergent modular forms after our construction of overconvergent modular forms is explained in Section 7.

In order to obtain optimal control over the torus action and the field of coefficients, we work harder in this section than is necessary for the proofs of Theorems A and B. This added control may be helpful in future applications to families of modular forms, and is useful in explaining the connection with Serre's mod p correspondence [26] (cf. 5.8.4).

5.1 Fixing the CM data

In our construction of evaluation maps, we will make a choice of a quadratic extension F/\mathbb{Q}_p , a point $x \in \Omega(F)$, and a point

$$x_\infty \in \pi_{\text{HT}}^{-1}(x) \subset \widehat{LT}^\infty(\widehat{F^{\text{ab}}}).$$

As explained in 3.4, we can package this information concretely as the choice of

- an order R in F , which determines a unique CM formal group G over $\mathcal{O}_{\check{F}_{R^\times}}$ with CM order R ,
- a trivialization

$$\text{triv} : \mathbb{Z}_p^2 \rightarrow T_p G,$$

- and a quasi-isogeny

$$\rho_0 : G_{0, \mathcal{O}_{\check{F}_{R^\times}}/p} \rightarrow G_{\mathcal{O}_{\check{F}_{R^\times}}/p}.$$

The most important aspects of these choices in our construction are captured by the classifying point $x \in \Omega(F)$. Given a choice of x , the possible choices of x_∞ form a $D^\times(\mathbb{Q}_p)$ -torsor (where $D^\times(\mathbb{Q}_p)$ acts by changing ρ_0).

We note that these choices determine embeddings

$$j_D : F \rightarrow D(\mathbb{Q}_p)$$

and

$$j_M : F \rightarrow M_2(\mathbb{Q}_p)$$

such that

$$\rho_0 \circ j_D(a) = a \circ \rho_0$$

and

$$\text{triv} \circ j_M(a) = a_* \circ \text{triv}.$$

5.1.1 Equivariant trivialization

As in 3.4.2, we find that $j_M(F^\times)$ preserves x , and over \mathbb{C}_p , there is a natural identification

$$\omega_G[1/p](-1) \xrightarrow{\sim} \Gamma(\mathcal{O}(1)|_x) \tag{5.1.1.1}$$

where, for $a \in F^\times$ the left action of a via j_M is identified with the action of $(a^{-1})^*$ on the top. For integers s and t , we consider the character

$$\begin{aligned} \tau_{s,t} : F^\times &\rightarrow F^\times \\ z &\mapsto z^s \bar{z}^t. \end{aligned}$$

Lemma 5.1.2. *The choice of a non-zero element $v \in \Gamma(\mathcal{O}(1)|_x)$ induces an F^\times -equivariant trivialization*

$$\tau_{-1,0} \otimes \mathcal{O}|_x \xrightarrow{\sim} \mathcal{O}(1)|_x.$$

Proof. This is just the statement that v transforms under the left action of F^\times through j_M via $\tau_{-1,0}$. It suffices to verify this after base-change to \mathbb{C}_p , where it follows from the identification through (5.1.1.1) of the left action of $j_M(a)$ with the action of $(a^{-1})^*$ on $\omega_G[1/p](-1)$, and our convention for the CM action (cf. Remark 3.4.5), which says this is given by a^{-1} . □

5.2 Moduli interpretation

Using the notation of 4.1.4, we let

$$\mathrm{Ig}_{K^p} := \mathrm{Ig}_{E_0[p^\infty], K^p, \mathcal{O}_{\check{F}_{R^\times}}}.$$

We give it the moduli interpretation of 4.1.5: for $R \in \mathrm{Comp}_R$, it parameterizes triples

$$(E, \rho, \alpha) \tag{5.2.0.1}$$

where $(E/R, \alpha)$ is an elliptic curve up to isogeny with level K^p -structure, and

$$\rho : G_R \rightarrow E[p^\infty]$$

is a quasi-isogeny.

The action of F^\times as the composition with j_D and the right action of $D^\times(\mathbb{Q}_p)$ on Ig_{K^p} has a natural interpretation in this moduli interpretation viewing $F = \mathrm{End}^0(G)$: for $a \in F^\times$

$$(E, \rho, \alpha) \cdot j_D(a) = (E, \rho \circ a, \alpha).$$

5.3 A reciprocity law

We denote X_{x, K^p} the fiber of the π_{HT} above x in X_{∞, K^p} (viewed as an adic space over \check{F}).

By Theorem 4.3.1, the point x_∞ determines an isomorphism

$$\mathrm{Ig}_{K^p, \widehat{F^{\mathrm{ab}}}}^{\mathrm{ad}} \rightarrow X_{x, K^p, \widehat{F^{\mathrm{ab}}}} \tag{5.3.0.1}$$

given on $\widehat{F^{\mathrm{ab}}}$ points by

$$(E/\mathcal{O}_{\widehat{F^{\mathrm{ab}}}}, \rho, \alpha) \mapsto (E, \rho_* \circ \mathrm{triv}, \alpha).$$

On the right-hand side of (5.3.0.1), we have an action of F^\times via j_M , as π_{HT} is $GL_2(\mathbb{Q}_p)$ -equivariant and x is stabilized by $j_M(F^\times)$. On the left-hand side we have an action of F^\times via j_D (cf. also the description in 5.2).

From 5.3.0.1, we also obtain two Galois actions of

$$R^\times = \text{Gal}_{F^{\text{ab}}, \check{F}/\check{F}}$$

on $\text{Ig}_{K^p, \widehat{F^{\text{ab}}}}^{\text{ad}}$. The first, σ_1 , comes from the rational structure $\text{Ig}_{K^p, \check{F}}^{\text{ad}}$. The second, σ_2 , comes from the rational structure $X_{x, K^p, \check{F}}$.

Thus, we have a total of four actions of R^\times on the same space. They are intertwined by the following reciprocity law:

Lemma 5.3.1 (Reciprocity law). *The isomorphism (5.3.0.1) identifies the action of F^\times via j_D on*

$$\text{Ig}_{K^p, \widehat{F^{\text{ab}}}}^{\text{ad}}$$

with the action of F^\times via j_M on

$$X_{x, K^p, \widehat{F^{\text{ab}}}}$$

Furthermore, the two Galois actions σ_1 and σ_2 of R^\times on

$$\text{Ig}_{K^p, \widehat{F^{\text{ab}}}}^{\text{ad}}$$

are related by $\sigma_2 = \sigma_1 \circ (\cdot j_D^{-1})$. where the -1 denotes inverse in D^\times .

Proof. The map (5.3.0.1) is determined by its action on $\widehat{F^{\text{ab}}}$ points, as described above.

For $a \in R^\times$, we have

$$(E, \rho, \alpha) \cdot j_D(a) = (E, \rho \circ a, \alpha)$$

which maps to

$$(E, \psi', \alpha)$$

where ψ' is

$$(\rho \circ a)_* \circ \text{triv} = \rho_* \circ \psi \circ j_M(a)$$

Thus we conclude the actions of $j_M(a)$ and $j_D(a)$ are intertwined by (5.3.0.1).

It remains to verify the Galois action. For a function

$$f \in H^0(X_{x, K^p, \widehat{F^{\text{ab}}}}, \mathcal{O}),$$

we have

$$(\sigma \cdot f)((E, \psi, \alpha)) = \sigma(f(\sigma^{-1} \cdot (E, \psi, \alpha)))$$

Since (E, α) is defined over \check{F}_{R^\times} , the Galois action only moves ψ . Moreover, it suffices to consider $\psi = \rho_* \circ \text{triv}$. The Galois action of σ^{-1} is by $a_{\sigma^{-1}*}$, thus

$$\sigma^{-1} \cdot (E, \psi, \alpha) = (E, \psi, \alpha) \cdot j_M(a_\sigma^{-1})$$

On the other hand, for a function

$$f \in H^0(\text{Ig}_{K^p, \widehat{F^{\text{ab}}}}^{\text{ad}}, \mathcal{O}),$$

since all points are defined over \check{F} , we have

$$(\sigma \cdot f)(y) = \sigma(f(y)).$$

Using the identification of the actions of j_D and j_M , we obtain

$$\sigma_2 = \sigma_1 \circ (\cdot j_D^{-1}).$$

□

5.4 Equivariant trivialization of ω

The map π_{HT} restricts to an $F^\times \times GL_2(\mathbb{A}_f^{(p)})$ -equivariant map (with F^\times acting through j_M)

$$\pi_{\text{HT}} : (X_{x,K^p})_{K^p} \rightarrow (x)_{K^p}$$

Recall that by Lemma 5.1.2, the choice of a non-zero $v \in \mathcal{O}(1)|_x$ induces an isomorphism of $F^\times \times GL_2(\mathbb{A}_f^{(p)})$ -equivariant bundles on $(x)_{K^p}$

$$\tau_{-1,0} \otimes \mathcal{O}|_x \rightarrow \mathcal{O}(1)|_x.$$

By Theorem 3.2.9, there is a canonical isomorphism of $GL_2(\mathbb{A}_f)$ -equivariant bundles on $(X_{\infty,K^p})_{K^p}$

$$\pi_{\text{HT}}^* (\det_p \cdot \det_{\text{ur}} \otimes \mathcal{O}(1)) \cong \omega.$$

Thus, we obtain an isomorphism of $F^\times \times GL_2(\mathbb{A}_f^{(p)})$ -equivariant vector bundles on $(X_{x,K^p})_{K^p}$

$$\tau_{-1,0} \cdot \det_p \cdot \det_{\text{ur}} \otimes \mathcal{O} \xrightarrow{\sim} \omega$$

Note that, $\tau_{-1,0} \cdot \det_p = \tau_{0,1}$, thus we can rewrite this as

$$\tau_{0,1} \cdot \det_{\text{ur}} \otimes \mathcal{O} \xrightarrow{\sim} \omega \tag{5.4.0.1}$$

(this is a more useful expression for us, since \det_{ur} is trivial on \mathcal{O}_F^\times).

5.5 A twisting function

There is a natural function

$$\det_{D^\times} : D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / D^\times(\widehat{\mathbb{Z}^{(p)}}) \rightarrow \mathbb{Z}_p^\times$$

Given as the composition of the reduced norm

$$\text{Nrd} : D^\times \rightarrow \mathbb{G}_m$$

composed with the p -adic cyclotomic character

$$\mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / \widehat{\mathbb{Z}}^{(p)\times} \rightarrow \mathbb{Z}_p^\times.$$

Concretely,

$$\det_{D^\times} \left(\prod g_l \right) = \text{Nrd}(g_p) | \text{Nrd}(g_p) |_p \prod_{l \neq p} | \text{Nrd}(g_l) |_l.$$

We note that under our isomorphism $\text{GL}_2(\mathbb{A}_f^{(p)}) \cong D^\times(\mathbb{A}_f^{(p)})$, the reduced norm is identified with determinant.

The function \det_{D^\times} gives rise to an isomorphism

$$(\tau_{1,1} \otimes \det_{\text{ur}}) \otimes \mathcal{O} \xrightarrow{\sim} \mathcal{O} \tag{5.5.0.1}$$

$$1 \mapsto \det_{D^\times}$$

of $F^\times \times \text{GL}_2(\mathbb{A}_f^{(p)})$ -equivariant bundles on

$$(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p)_{K^p}^{\text{ad}} = \text{Ig}_{K^p}^{\text{ad}}.$$

5.6 Restricting modular forms

Lemma 5.6.1. *Fix a $K_p \subset \text{GL}_2(\mathbb{Q}_p)$ and $k \in \mathbb{N}$. The maps*

$$H^0(X_{K_p K^p}, \omega^k) \otimes \check{F}_{R_{K^p}^\times} \rightarrow H^0(X_{x, K^p, \widehat{F}^{\text{ab}}}, \omega^k)$$

are injective and map into the set of Galois invariants for the Galois action of $R_{K^p}^\times$.

Proof. The image lies in the Galois invariants because everything is defined over $\check{F}_{R_{K^p}^\times}$. To show the maps are injective, we observe that the image of X_{x,K^p} in $X_{K^p K^p}$ contains infinitely many $\widehat{F^{\text{ab}}}$ points in each component – indeed, if we work over $\widehat{F^{\text{ab}}}$, then under the isomorphism

$$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p \xrightarrow{\sim} \text{Ig}_{K^p},$$

the map factors through an injection from

$$D^\times(\mathbb{Q}_p) \backslash D^\times(\mathbb{A}_f) / R_{K^p}^\times K^p$$

and the components correspond to values of \det_{D^\times} . Thus, any section from finite level which vanishes along X_{x,K^p} must be identically zero. \square

5.7 Evaluating modular forms

Taking the k th power of (5.4.0.1) and tensoring with the isomorphism (3.2.6.2), we obtain an isomorphism of $F^\times \times GL_2(\mathbb{A}_f^{(p)})$ -equivariant bundles on X_{x,K^p}

$$\tau_{0,k} \cdot \det_{\text{ur}}^{k-t} \mathcal{O} \xrightarrow{\sim} \omega^k \otimes \det_{\text{dR}}^t \quad (5.7.0.1)$$

We base change to $\widehat{F^{\text{ab}}}$ and pullback via 5.3.0.1 to obtain an isomorphism over

$$(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p)_{\widehat{F^{\text{ab}}}}^{\text{ad}} \cong \text{Ig}_{K^p \widehat{F^{\text{ab}}}}^{\text{ad}}.$$

By composing the map of Lemma 5.6.1 with the global sections of (5.4.0.1), we obtain Hecke and F^\times -equivariant injections

$$H^0(X_{K^p K^p}, \omega^k \otimes \det_{\text{dR}}^t) \otimes \check{F}_{R_{K^p}^\times} \hookrightarrow \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \widehat{F^{\text{ab}}}) \otimes \tau_{0,k} \cdot \det_{\text{ur}}^{k-t}. \quad (5.7.0.2)$$

Now, the left-hand side has the trivial action of $R_{K_p}^\times$ acting both through F^\times and as the Galois group. Thus, this map factors through the invariants on the right for the action of $R_{K_p}^\times$ both through F^\times and through the twisted Galois action σ_2 . The invariants for the action of $R_{K_p}^\times$ through F^\times are naturally identified with the isotypic component of

$$\text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \widehat{F^{\text{ab}}}) \otimes \det_{\text{ur}}^{k-t}$$

where $R_{K_p}^\times$ acts through j_D by the character \bar{z}^{-k} . Now, by Lemma 5.3.1,

$$\sigma_2 = \sigma_1 \circ \cdot j_D^{-1},$$

where σ_1 acts only on the coefficients, and thus we conclude that the functions take values in the isotypic component of $\widehat{F^{\text{ab}}}$ where $R_{K_p}^\times$ acts via Galois as \bar{z}^{-k} . This is a one-dimensional $\check{F}_{R_{K_p}^\times}$ -vector space, and we may choose a basis to identify it with $\check{F}_{R_{K_p}^\times}$ (cf. Remark 5.7.1 below). Thus, we obtain a F^\times , Hecke equivariant map

$$H^0(X_{K_p K^p}, \omega^k \otimes \det_{\text{dR}}^t) \otimes \check{F}_{R_{K_p}^\times} \hookrightarrow \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \check{F}_{R_{K_p}^\times}) \otimes \tau_{0,k} \cdot \det_{\text{ur}}^{k-t}. \quad (5.7.0.3)$$

Remark 5.7.1. Choose non-zero elements $\partial \in \text{Lie}G$ and $v \in T_p G^\vee$. The element v induces a map $G \rightarrow \widehat{\mathbb{G}}_m$ over $\mathcal{O}_{\widehat{F^{\text{ab}}}}$, and we can form the Hodge-Tate period

$$c_{v,\partial} := \langle \partial, v^* \frac{dt}{t} \rangle.$$

Then $c_{v,\partial}^k$ spans the \bar{z}^k isotypic component of $\widehat{F^{\text{ab}}}$.

We highlight here that the existence of an element $c_{v,\partial}$ in $\widehat{F^{\text{ab}}}$ transforming via \bar{a} under the Galois action of $a \in R^\times$ is somewhat surprising: in taking the completion $\widehat{F^{\text{ab}}}$, it is not obvious that any non-smooth locally analytic vectors for the Galois action should appear!

Compare, for example, with $\widehat{\mathbb{Q}_p(\mu_p^\infty)}$ – the lack of locally analytic vectors for the \mathbb{Z}_p^\times action on this space is at the heart of Sen-Tate theory – see [2] for a detailed discussion of this phenomenon.

Using (5.5.0.1), we have a F^\times and Hecke-equivariant identification of the right-hand side of (5.7.0.3) with

$$\mathrm{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \check{F}_{R_{K_p}^\times}) \otimes \tau_{t-k,t}.$$

Furthermore, the map 5.7.0.3 factors through the $R_{K_p}^\times$ -invariants for the F^\times action, which can now identify with

$$\mathrm{Hom}_{R_{K_p}^\times} \left(\tau_{k-t,-t}, \mathrm{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \check{F}_{R_{K_p}^\times}) \right).$$

Thus, we have constructed evaluation maps

$$\begin{aligned} \mathrm{eval}_x^{k,t,K_p,K^p} : H^0(X_{K_p K^p}, \omega^k \otimes \det_{\mathrm{dR}}^t) \otimes \check{F}_{R_{K_p}^\times} \\ \hookrightarrow \mathrm{Hom}_{R_{K_p}^\times} \left(\tau_{k-t,-t}, \mathrm{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \check{F}_{R_{K_p}^\times}) \right), \end{aligned} \quad (5.7.1.1)$$

and we have shown

Theorem 5.7.2. *The maps $\mathrm{eval}_x^{k,t,K_p,K^p}$ are Hecke-equivariant injections.*

Remark 5.7.3. If we change t , the underlying space on the left stays the same; only the Hecke action is changed. The maps for different t then differ via multiplication by the function \det_{D^\times} of 5.5.

5.8 Two classical interpretations

In this section we give classical interpretations of the evaluation maps (5.7.1.1) when $t = k$ and when $t = 0$. In particular, we explain how our evaluation maps generalize the mod p evaluation maps of Serre [26].

The basic idea is the following: combining the second moduli interpretation of the Igusa formal scheme as in 4.1.5 with Theorem 4.2.1, we find that

$$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p$$

as a formal scheme over $\mathrm{Spf}_{\mathcal{O}_{\check{F}_R}}$ can be interpreted as a moduli space of triples

$$(E, \rho, \alpha)$$

where E is an elliptic curve, α is a level K^p structure, and

$$\rho : G \xrightarrow{\sim} E[p^\infty].$$

Under this interpretation, we have for $a \in R^\times$,

$$(E, \rho, \alpha) j_D(a) = (E, \rho \circ a, \alpha).$$

Let $R_{K_p}^\times = j_M(F) \cap GL_2(\mathbb{Z}_p)$. Then the K_p -orbit of triv gives a level K_p structure defined over $\check{F}_{R_{K_p}^\times}$, so that we obtain a classifying map

$$(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p)_{\check{F}_{R_{K_p}^\times}}^{\mathrm{ad}} \rightarrow X_{K_p K^p, \check{F}_{R_{K_p}^\times}} \quad (5.8.0.1)$$

which factors through an injection

$$(D^\times(\mathbb{Q}) \backslash D_{\mathbb{A}_f}^\times / j_D(R_{K_p}^\times) K^p)_{\check{F}_{R_{K_p}^\times}}^{\mathrm{ad}} \hookrightarrow X_{K_p K^p, \check{F}_{R_{K_p}^\times}}. \quad (5.8.0.2)$$

Thus, we may pull back sections of $H^0(X_{K_p K^p, \check{F}_{R_{K_p}^\times}}, \omega^k)$ to the double coset. To obtain functions, we evaluate these functions using two natural trivializations of ω over the double coset.

Remark 5.8.1. At this point, rather than working with adic spaces we could work with schemes by taking Spec of the continuous functions on the double coset.

5.8.2 The Serre trivialization

The isomorphism $\rho : G \rightarrow E[p^\infty]$ induces an isomorphism between the trivial bundle $\text{Lie}G \otimes \mathcal{O}$ and ω^{-1} over

$$(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p)^{\text{ad}}_{\check{F}_{R_{K^p}^\times}}.$$

Thus, given a basis of $\text{Lie}G$, we obtain a basis $\rho_*\partial$ of ω^{-1} , which we will denote simply by ∂ . Pulling back modular forms via 5.8.0.1, we obtain evaluation maps

$$H^0(X_{K_p K^p}, \omega^k) \otimes \check{F}_{R_{K^p}^\times} \rightarrow \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \check{F}_{R_{K^p}^\times}) \quad (5.8.2.1)$$

$$g \mapsto \langle (\rho_*\partial)^k, g \rangle. \quad (5.8.2.2)$$

We observe that since

$$(E, \rho, \alpha) \cdot j_D(a) = (E, \rho \circ a, \alpha),$$

we have

$$j_D(a) \cdot \rho_*\partial = \rho_*(a_*\partial) = \rho_*(a\partial) = a\rho_*\partial.$$

On the other hand, by the factorization 5.8.0.2, we see that for $a \in R_{K^p}^\times$, $j_D(a) \cdot g = g$.

Thus, for such an a ,

$$j_D(a) \cdot \langle (\rho_*\partial)^k, g \rangle = \langle (j_D(a) \cdot \rho_*\partial)^k, j_D(a) \cdot g \rangle = \langle a^k \rho_*\partial, g \rangle = a^k \langle \rho_*\partial, g \rangle.$$

Thus, the image of 5.8.2.2 lies in the a^k character space for the action of $R_{K^p}^\times$ via j_D , and indeed, this is the map

$$\text{eval}_x^{k,0,K_p,K^p}.$$

5.8.3 The Tate trivialization

We fix a non-zero element $v \in T_p G$, which can be defined over $\widehat{F^{\text{ab}}}$. On the universal (E, ρ) over

$$(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p) \mathcal{O}_{\widehat{F^{\text{ab}}}},$$

we obtain the map induced by the Weil pairing

$$\langle \cdot, \rho_* v \rangle_E : \widehat{E} \rightarrow \widehat{\mathbb{G}}_m.$$

We thus obtain

$$\eta_v := (\langle \cdot, \rho_*(v) \rangle_E)^* \frac{dt}{t}.$$

After passing to the generic fiber, η_v is a basis ω (as is typical for p-adic Hodge theory outside of the ordinary case, there is a torsion cokernel in the relevant integral comparison), and we obtain an evaluation map

$$H^0(X_{K_p K^p, \check{F}_{R_{K_p}^\times}}, \omega^k) \rightarrow \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \widehat{F^{\text{ab}}})$$

We observe that for $a \in R_{K_p}^\times$,

$$\langle \cdot, \rho_* a_*(v) \rangle_E = \langle \bar{a} \cdot, \rho_* \rangle_E = \langle \cdot, \rho_*(v) \rangle_E \circ \bar{a}.$$

Thus,

$$j_D(a) \cdot \eta_v = (\langle \cdot, \rho_*(v) \rangle_E \circ \bar{a})^* \frac{dt}{t} = \bar{a}^* (\langle \cdot, \rho_*(v) \rangle_E)^* \frac{dt}{t} = \bar{a} \eta_v.$$

So, the resulting functions are in the \bar{a}^{-k} character space for the action of $R_{K_p}^\times$ through j_D . Similarly we find $\sigma \eta_v = \bar{a} \eta_v$. Since the identification of $R_{K_p}^\times$ with the Galois group is via the action on the Tate module, we find that the Galois action on the values of the functions in the image is via \bar{a}^{-k} , arguing as in the proof of Lemma 5.3.1. If we divide by the k th

power of a period, we obtain the map

$$\text{eval}_x^{k,k,K_p,K^p}.$$

5.8.4 Integral evaluation and Serre's mod p Jacquet-Langlands

We consider the concrete instance of 5.8.2 given by taking $F = \mathbb{Q}_{p^2}$, the unramified quadratic extension of \mathbb{Q}_p , $R = \mathbb{Z}_{p^2}$, the ring of integers, and $K_p = \text{GL}_2(\mathbb{Z}_p)$. In this case, $(X_{\text{GL}_2(\mathbb{Z}_p)K^p}, \omega)$ has a natural smooth formal model $(\mathfrak{X}_{K^p}, \mathfrak{w})$ over $\check{\mathbb{Z}}_p$. We can refine the Serre evaluation maps (5.8.2.2) (which do not depend at all on the trivialization of $T_p G$) to maps

$$H^0(\mathfrak{X}_{K^p}, \mathfrak{w}^k) \hookrightarrow \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \check{\mathbb{Z}}_p) \quad (5.8.4.1)$$

landing in the isotypic component where \mathbb{Z}_{p^2} acts by a^k .

Let U_1 denote the kernel of reduction modulo the uniformizer in $\mathcal{O}_{D(\mathbb{Q}_p)}^\times$. In [26], Serre constructs an evaluation map¹

$$\text{eval}_{\text{Serre}} : H^0(\mathfrak{X}_{K^p, \overline{\mathbb{F}}_p}, \mathfrak{w}_{\overline{\mathbb{F}}_p}^k) \rightarrow \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / U_1 K^p, \overline{\mathbb{F}}_p)$$

which lands in the a^k character space (which still makes sense since U_1 is normal and $a^k \bmod p$ is trivial on $U_1 \cap \mathbb{Z}_{p^2}^\times$). Comparing with the construction of [26], we find

Theorem 5.8.5. *The mod p maps induced from (5.8.4.1) factor as*

$$H^0(\mathfrak{X}_{K^p}, \mathfrak{w}^k) / p \rightarrow H^0(\mathfrak{X}_{K^p, \overline{\mathbb{F}}_p}, \mathfrak{w}_{\overline{\mathbb{F}}_p}^k) \xrightarrow{\text{eval}_{\text{Serre}}} \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / U_1 K^p, \overline{\mathbb{F}}_p) \hookrightarrow \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \overline{\mathbb{F}}_p).$$

1. There are some minor differences between our conventions and those of [26] – for example, Serre writes the double coset with adelic quotient on the left and the rational quotient on the right.

CHAPTER 6

AN ISOMORPHISM OF HECKE ALGEBRAS

In this section we prove Theorem A. Our strategy is to use the evaluation maps of Section 5 in order to transfer modular forms to quaternionic functions, then to use the technique of fake Hasse invariants to show that any function on the quaternionic double coset can be arbitrarily well approximated by classical modular forms. For these purposes we may work over \mathbb{C}_p and without worrying about the division algebra action, which leads to some simplifications in the construction of the evaluation maps.

In 6.1.3 we describe the completion process for algebras acting on families of Banach spaces and provide some useful lemmas for comparing completions. In 6.2 we define the completed Hecke algebras appearing in Theorem A. In 6.3 we explain the simplified evaluation maps, and in 6.4 we show their image is dense. Finally, in 6.5 we combine these ingredients to prove Theorem A.

6.1 Completing actions

We introduce some functional analysis which will be useful for defining and comparing the completed Hecke algebras appearing in Theorem A. These results are likely well-known, but we were unable to find a suitable reference.

We refer the reader to the introduction of [27] for the basic definitions and results on Banach spaces over non-archimedean fields.

6.1.1 Strong completion

Definition 6.1.2. An action of a (not necessarily commutative) ring A by bounded operators on a Banach space V is *uniform* if for all $a \in A$ and $v \in V$,

$$\|a \cdot v\| \leq \|v\|.$$

Definition 6.1.3. If A is a ring, K is a non-archimedean field, and (W_i) is family of Banach spaces equipped with uniform actions of A , the *strong completion* of A with respect to $(W_i)_{i \in I}$ is the closure \widehat{A} of the image of A in

$$\prod_{i \in I} \text{End}_{\text{cont}}(W_i)$$

where each $\text{End}_{\text{cont}}(W_i)$ is equipped with the strong operator topology (the topology of pointwise convergence for the strong topology on W_i) and the product is equipped with the product topology.

We give two equivalent characterizations of the elements of \widehat{A} :

Lemma 6.1.4. *In the setting of Definition 6.1.3:*

(Nets) $\prod_i f_i \in \widehat{A}$ if and only if there exists a net $a_j \in A$ such that for any $i \in I$ and any $w \in W_i$,

$$\lim a_j \cdot w = f_i(w).$$

(Congruences) For each $i \in I$, fix a choice W_i° of a lattice in W_i preserved by A (e.g., the elements of norm ≤ 1). Then, $\prod_i f_i \in \widehat{A}$ if and only if f_i preserves W_i for each i , and for any finite subset $S \subset I$ and any topologically nilpotent $\pi \in K$, there exists $a \in A$ such that for each $i \in S$, a and f_i have the same image in

$$\text{End}(W_i^\circ/\pi).$$

Proof. The characterization (Nets) is immediate from the definition of the strong operator topology as the topology of pointwise convergence of nets and the characterization of the product topology as the topology of term-wise convergence of nets.

The characterization (Congruences) then follows by considering nets on the directed set of finite subsets of I times \mathbb{N} (where \mathbb{N} is interpreted as the power of some fixed uniformizer)

to show that (Congruences) implies (Nets). \square

Using either the characterization in terms of nets plus uniformity of the action, or the characterization in terms of congruences, we find that \widehat{A} is again a ring. It is equipped with a natural structure as an A -algebra.

Remark 6.1.5. By (Congruences), we can also construct \widehat{A} as the closure of the image of A in

$$\prod \text{End}(W_i^\circ/\pi)$$

where the product is over all possible choices of $i \in I$, a lattice $W_i^\circ \subset W_i$, and a topologically nilpotent π , and each term has the discrete topology.

6.1.6 Relating strong completions

In order to compare completed Hecke algebras, we will need some lemmas.

The following lemma says that formation of the strong completion is insensitive to base extension. This will be useful for us as our comparisons of Hecke-modules take place over large extensions of \mathbb{Q}_p , whereas we are interested in Hecke algebras over \mathbb{Z}_p .

Lemma 6.1.7. *Let $K \subset K'$ be an extension of complete non-archimedean fields, and let A be a (not-necessarily commutative) ring. Suppose (W_i) is a family of orthonormalizable Banach spaces over K equipped with uniform actions of A . Then the identity map $A \rightarrow A$ extends to a topological isomorphism between the strong completions of A acting on (W_i) and A acting on $(W_i \widehat{\otimes}_K K')$.*

Proof. We note that for a bounded net ϕ_j of bounded operators on a orthonormalizable Banach space, $\phi_j \rightarrow f$ in the strong operator topology if and only if $\phi_j(e) \rightarrow f(e)$ for any element e of a fixed orthonormal basis.

In particular, because an orthonormal basis for W_i is also an orthonormal basis for W_i' , we find that the strong completion for (W_i) injects into the strong completion for $(W_i \widehat{\otimes}_K K')$.

More over, since W_i is closed inside of $W_i \widehat{\otimes}_K K'$ and preserved by A , we find that for any net $a_j \in \mathbb{T}'$ and element e in the orthonormal basis, $\lim_j a_j(e)$ is in W_i if it exists. Thus, an element in the strong completion for $(W_i \widehat{\otimes}_K K')$ comes from an element in the strong completion for (W_i) . \square

The following lemma is our main technical tool. It says that the strong completion is determined by any family of invariant subspaces whose sum is dense.

Lemma 6.1.8. *Let K be a non-archimedean field, and let A be a (not-necessarily commutative) ring. Suppose V is an orthonormalizable K -Banach space equipped with a uniform action of A , and $(W_i)_{i \in I}$ is a family of topological vector spaces over K equipped with A -actions and continuous A -equivariant topological immersions*

$$\psi_i : W_i \hookrightarrow V.$$

If $\sum \text{Im} \psi_i$ is dense in V , then the identity map on A induces an isomorphism between the weak completion of A' acting on $(W_i)_{i \in I}$ and the weak completion of A acting on V .

Remark 6.1.9. In this setup, the action of A on W_i is automatically uniform for the restriction to W_i of the norm on V , which, by hypothesis, induces the same topology.

Proof. Denote by $\widehat{A}_V \subset \text{End}(V)$ the strong completion of A acting on V , and $\widehat{A}_W \subset \prod \text{End}(W_i)$ the strong completion of A acting on $(W_i)_{i \in I}$.

We first show there is a map $\widehat{A}_V \rightarrow \widehat{A}_W$ extending the identity map $A \rightarrow A$: Let $\phi \in \widehat{A}_V$, and let ϕ_j be a net in the image of A approaching ϕ . For $w \in W_i$ (considered as closed subspace of V via ψ_i),

$$\phi(w) = \lim_j \phi_j(w).$$

For each j , $\phi_j(w)$ is contained in W_i by the A -equivariance of ψ_i , and thus, since W_i is

closed, $\phi(w) \in W_i$. Thus, ϕ preserves W_i . Using this, we obtain a map

$$\widehat{A}_V \rightarrow \prod_i \text{End}(W_i)$$

extending the map $A \rightarrow \prod_i \text{End}(W_i)$. Furthermore, it follows immediately that the image lies in \widehat{A}_W .

The map is injective by the density of $\sum W_i \subset V$. We show now that it is surjective. By the density of $\sum \text{Im} \psi_i$, we may choose an orthonormal basis for V contained in the image of $\oplus W_i$. A bounded net of operators in $\text{End}(V)$ converges if and only if it converges on each element of an orthonormal basis. Now, if $\phi \in \widehat{A}_W$ is the limit of a net ϕ_j in the image of A , then we see that $\phi_j(e)$ converges for each element e of the orthonormal basis, and thus ϕ_j also converges in $\text{End}(V)$, and its limit maps to ϕ , as desired.

Thus the map $\widehat{A}_V \rightarrow \widehat{A}_W$ is bijective. By similar arguments, the weak topologies agree, and thus the map is a topological isomorphism. \square

As a special case, we obtain an alternative description of the strong completion in some cases:

Lemma 6.1.10. *Let V be an orthonormalizable Banach space over K equipped with a uniform action of an \mathcal{O}_K -algebra A . Suppose (W_i) is a directed system of finite dimensional K -vector spaces with A actions and compatible maps $W_i \rightarrow V$. Suppose further that the maps $W_i \rightarrow V$ and the transition maps are A -equivariant and injective. Let A_i denote the image of A in $\text{End}(W_i)$, equipped with its natural topology ($\text{End}(W_i)$ is a finite dimensional K -vector space).*

If the image of

$$\varinjlim W_i \rightarrow W$$

is dense, then, the strong completion of \widehat{A} acting on V is equal to

$$\varprojlim A_i.$$

Proof. Applying Lemma 6.1.8, it suffices to verify that $\varprojlim A_i$ is the strong completion of A with respect to (W_i) . This is clear, as the image of A in

$$\prod \text{End}(W_i)$$

lies within and is dense in $\varprojlim A_i$, which is easily seen to be closed (in the strong operator topology). \square

6.2 Some completed Hecke algebras

We fix a compact open $K^p \subset GL_2(\mathbb{A}_f^{(p)})$, and let

$$\mathbb{T}_{\text{abs}} = \mathbb{Z}_p[GL_2(\mathbb{A}_f^{(p)})//K^p].$$

be the abstract Hecke algebra of prime-to- p level K^p . For any \mathbb{Z}_p sub-algebra $\mathbb{T}' \subset \mathbb{T}_{\text{abs}}$, we form the following strong completions:

- \mathbb{T}'_{D^\times} is the strong completion of \mathbb{T}' acting on

$$\text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K^p, \mathbb{Q}_p)$$

- $\mathbb{T}'_{\text{GL}_2}$ is the strong completion of \mathbb{T}' acting on $\widehat{H}_{K^p}^1$, the completed cohomology of the tower of modular curves (cf. [9]).
- \mathbb{T}'_{mf} is the strong completion of \mathbb{T}' acting on the space of Katz p -adic modular functions (cf. [15]).

Remark 6.2.1. We provide some alternate descriptions of these completed Hecke algebras, which are taken as the definition in other sources:

- For $K_p \subset D^\times(\mathbb{Q}_p)$, denote by $\mathbb{T}'_{D^\times, K_p}$ the image of \mathbb{T}' in

$$\text{End}_{\mathbb{Q}_p}(\text{Cont}(D^\times(\mathbb{Q}_p) \backslash D^\times(\mathbb{A}_f) / K_p K^p, \mathbb{Q}_p)).$$

Applying Lemma 6.1.10, we find

$$\mathbb{T}'_{D^\times} = \varprojlim \mathbb{T}'_{D^\times, K_p}.$$

- For $K_p \subset GL_2(\mathbb{Q}_p)$, denote by \mathbb{T}'_{GL_2, K_p} the image of \mathbb{T}' in $\text{End}_{\mathbb{Q}_p}(H^1(Y_{K_p K^p}, \mathbb{Q}_p))$.

Applying Lemma 6.1.10, we find

$$\mathbb{T}'_{GL_2} = \varprojlim \mathbb{T}'_{GL_2, K_p};$$

cf. [10, Definition 5.2.1 and p46, footnote 12].

- For $n \in \mathbb{N}$, let

$$M_n = \oplus_{k \leq n} H^0(X_{GL_2(\mathbb{Z}_p)K^p}, \omega^k \otimes \det_{\text{dR}}^{-1}),$$

and let \mathbb{T}'_n be the image of \mathbb{T}' in $\text{End}(M_n)$. We may view M_n as a subspace of the ring $\mathbb{V}_{\text{Katz}} \otimes \mathbb{Q}_p$ of Katz p -adic modular functions, and the sum of the M_n is dense.

Applying Lemma 6.1.10, we find

$$\mathbb{T}'_{\text{mf}} = \varprojlim \mathbb{T}'_n.$$

6.3 The comparison maps

To prove an isomorphism of Hecke algebras, we are free to work over \mathbb{C}_p (by Lemma 6.1.7) and to forget about the action $D^\times(\mathbb{Q}_p)$. This leads to two simplifications in the construction of the evaluation maps of Section 5:

- Instead of trivializing $\mathbb{Q}_p(-1)$ via the Weil pairing, we may simply fix a compatible system of roots of unity in \mathbb{C}_p .
- We may work over any point $x \in \Omega(\mathbb{C}_p)$ rather than a point over a quadratic extension of \mathbb{Q}_p (i.e., we may work with any height 2 formal group instead of a CM height 2 formal group).

We now describe the evaluation maps we will use. Let $x \in \Omega(\mathbb{C}_p)$ and fix $x_\infty \in \widehat{\text{LT}}^\infty(\mathbb{C}_p)$ such that $\pi_{\text{HT}}(x_\infty) = x$. We denote by $X_{\infty K^p, x}$ the fiber of π_{HT} over x in $X_{\infty K^p, \mathbb{C}_p}$.

By Theorem 4.3.1, this choice induces an isomorphism

$$\text{Ig}_{K^p, \mathbb{C}_p}^{\text{ad}} \cong X_{\infty K^p, x},$$

and by Theorem 4.2.1, our initial choice of $E_0/\overline{\mathbb{F}_p}$ with trivialization of prime-to- p Tate module induces an isomorphism

$$(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K^p)_{\mathbb{C}_p}^{\text{ad}} \cong \text{Ig}_{K^p, \mathbb{C}_p}^{\text{ad}}.$$

The compositions of these isomorphisms fit into a $GL_2(\mathbb{A}_f^{(p)})$ -equivariant isomorphism of towers

$$\left((D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K^p)_{\mathbb{C}_p}^{\text{ad}} \right)_{K^p} \cong (X_{\infty K^p, x})_{K^p}.$$

By Theorem 3.2.9, we obtain an isomorphism of $GL_2(\mathbb{A}_f^{(p)})$ -equivariant bundles on $(X_{\infty K^p, \mathbb{C}_p})_{K^p}$

$$\pi_{\text{HT}}^*(\mathcal{O}(k)) \cong \omega^k(-k)$$

and, using our fixed compatible system of roots of unity in \mathbb{C}_p to trivialize $\mathbb{Q}_p(-1)$, we obtain an isomorphism of equivariant bundles

$$\pi_{\text{HT}}^*(\mathcal{O}(k)) = \omega^k$$

We now fix a nonzero $v \in \mathcal{O}(1)|_x$. This induces a $GL_2(\mathbb{A}_F^{(p)})$ -equivariant trivialization v^k of $(\mathcal{O}(k)|_x)_{K^p}$, and thus, via pullback, of ω^k restricted to $(X_{\infty K^p, x})_{K^p}$. The function \det_{D^\times} of 5.5 gives has a $GL_2(\mathbb{A}_f^{(p)})$ equivariant trivialization of \det_{dR} over $(\mathrm{Ig}_{K^p, \mathbb{C}_p}^{\mathrm{ad}})_{K^p}$, thus we obtain a $GL_2(\mathbb{A}_f^{(p)})$ -equivariant isomorphism of bundles

$$\mathcal{O} \xrightarrow{\sim} \omega^k \otimes \det_{\mathrm{dR}}.$$

This induces, for any K^p , a Hecke-equivariant evaluation isomorphism

$$\mathrm{eval}_x'^{k, K^p} : H^0(X_{\infty K^p, x}, \omega^k \otimes \det_{\mathrm{dR}}) \xrightarrow{\sim} \mathrm{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \mathbb{C}_p). \quad (6.3.0.1)$$

For any finite level K_p , we may compose with the Hecke-equivariant injection

$$H^0(X_{K_p K^p, \mathbb{C}_p}, \omega^k \otimes \det_{\mathrm{dR}}) \hookrightarrow H^0(X_{\infty K^p, x}, \omega^k \otimes \det_{\mathrm{dR}}^{-1}).$$

To obtain a Hecke-equivariant injection

$$H^0(X_{K_p K^p, \mathbb{C}_p}, \omega^k \otimes \det_{\mathrm{dR}}) \hookrightarrow \mathrm{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \mathbb{C}_p). \quad (6.3.0.2)$$

In order to deduce an isomorphism

$$\mathbb{T}'_{\mathrm{aux}} \rightarrow \mathbb{T}'_{D^\times}$$

of Hecke algebras from Lemma 6.1.8, we will show in the next section that the span of the images of these evaluation maps is dense (in fact, to make the argument concrete we will work with a specific choice of x , which is sufficient for our purposes).

6.4 Density of the evaluation maps

Let $\tau \in \mathbb{C}_p$ such that $|\tau| = 1$ and consider the point $x := [1, \tau] \in \Omega(\mathbb{C}_p)$. We choose an arbitrary element $x_\infty \in \widehat{\text{LT}}_\infty(\mathbb{C}_p)$ lying above x . We choose our non-zero element $v \in \mathcal{O}(1)|_x$ so that v^2 is the image of the global section XY of $\mathcal{O}(2)$. Using this data, we define the evaluation maps $\text{eval}_x'^{k, K^p}$ as in (6.3.0.1).

In this section we prove the following approximation lemma

Lemma 6.4.1. *If*

$$f \in \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \mathcal{O}_{\mathbb{C}_p}),$$

and $n > 0$, there exists a compact open $K_p \subset GL_2(\mathbb{Q}_p)$, a $k > 0$, and an

$$\omega_f \in H^0(X_{K_p K^p}, \omega^k \otimes \det_{\text{dR}})$$

such that

- $\text{eval}_x'^{k, K^p}(\omega_f) \in \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \mathcal{O}_{\mathbb{C}_p})$, and
- $\text{eval}_x'^{k, K^p}(\omega_f) \equiv f \pmod{p^n}$.

Proof. We are looking for ω_f that, when restricted to the fiber $X_{\infty K^p, x}$ and divided by $v^k(\det_{D^\times})$, gives an integral function reducing to $f \pmod{p^n}$. Since we may always replace f by $f \cdot \det_{D^\times}^{-1}$, we will simplify the problem by looking for ω_f that, when restricted and divided by v^k , gives an integral function reducing to f .

The proof consists of two steps: we first observe that f , considered as a function on $X_{\infty K^p, x}$, can be extended to an integral function on the pre-image of a small neighborhood of x . The second step shows that this function is the reduction of a modular form at some finite level using the technique of fake Hasse invariants as in [24, Proof of Theorem IV.3.1].

Let X and Y be the standard basis of global sections of $\mathcal{O}(1)$. Inside the affinoid

$$\left| \frac{Y}{X} \right| = 1 \subset \mathbb{P}^1,$$

we consider for $m \in \mathbb{N}$ the affinoid ball B_m containing x defined by $|\frac{Y}{X} - \tau| \leq |p^m|$.

Now, $\pi_{\text{HT}}^{-1}(B_1)$ is affinoid perfectoid as it is a rational subdomain of $|Y| = |X|$, which is affinoid perfectoid by Theorem 3.2.11. Because $X_{\infty K^p, x}$ is a Zariski closed subset defined on this affinoid by the equation

$$\frac{Y}{X} = \tau,$$

we may apply [24, Lemma II.2.2] to deduce that

$$H^0(\pi_{\text{HT}}^{-1}(B_1), \mathcal{O})$$

is dense in

$$H^0(X_{\infty K^p, x}, \mathcal{O}).$$

Thus we find

$$\tilde{f} \in H^0(\pi_{\text{HT}}^{-1}(B_1), \mathcal{O})$$

such that

$$\tilde{f}|_{X_{\infty K^p, x}} \in H^0(X_{\infty K^p, x}, \mathcal{O}^+)$$

and

$$\tilde{f}|_{X_{\infty K^p, x}} \pmod{p^n} = f.$$

We now want to see that the restriction of \tilde{f} to $\pi_{\text{HT}}^{-1}(B_m)$ is integral for m sufficiently large.

Let W be the rational open defined by $|\tilde{f}| \leq 1$ inside $\pi_{\text{HT}}^{-1}(B_1)$. Because f is integral, we have $X_{\infty K^p, x} \subset W$. Then, because

$$\bigcap_m \pi_{\text{HT}}^{-1}(\overline{B_m}) = X_{\infty K^p, x} \subset W$$

and $\pi_{\text{HT}}^{-1}(B_1) \setminus W$ is quasi-compact (it is closed inside $\pi_{\text{HT}}^{-1}(B_1)$, which is quasi-compact

because it is an affinoid), we find that for m sufficiently large,

$$\pi_{\text{HT}}^{-1}(B_m) \subset \pi_{\text{HT}}^{-1}(\overline{B_m}) \subset W$$

and thus

$$\tilde{f} \in H^0(\pi_{\text{HT}}^{-1}(B_m), \mathcal{O}^+).$$

We fix such an m , and consider the cover of \mathbb{P}^1 by the set B_m and the rational opens

$$U_1 := \left\{ \left| \frac{Y}{X} \right| \leq 1 \text{ and } \left| \frac{Y}{X} - \tau \right| \geq |p^m| \right\}, \quad U_2 := \left\{ \left| \frac{X}{Y} \right| \leq 1 \text{ and } \left| 1 - \frac{X}{Y} \tau \right| \geq |p^m| \right\}.$$

Remark 6.4.2. The reason for using three sets here rather than just B_m and a complementary set is to ensure that the pre-image of each set in the cover in $X_{\infty K^p, \mathbb{C}_p}$ is affinoid perfectoid so that sections can be approximated at finite level.

If we consider the sections of $\mathcal{O}(2)$

$$s_1 = p^{-m} X \cdot (Y - \tau X), \quad s_2 = p^{-m} Y \cdot (Y - \tau X) \text{ and } s_3 = XY,$$

then B_m is defined by the equations

$$|s_1/s_3| \leq 1 \text{ and } |s_2/s_3| \leq 1.$$

i.e.

$$|(Y - \tau X)/Y| \leq |p^m| \text{ and } |(Y - \tau X)/X| \leq |p^m|.$$

Furthermore, within B_m , $U_1 \cap B_m = U_2 \cap B_m$ is defined by either $|s_1/s_3| = 1$ or $|s_2/s_3| = 1$.

U_1 is defined by the equations

$$|s_2/s_1| \leq 1 \text{ and } |s_3/s_1| \leq 1.$$

Indeed, the first equation simplifies to

$$|Y/X| \leq 1$$

and the second equation simplifies to

$$\left| p^m \frac{\frac{Y}{X}}{\frac{Y}{X} - \tau} \right| \leq 1$$

In the presence of the first equation, this is equivalent to

$$\left| \frac{Y}{X} - \tau \right| \geq |p^m|.$$

Furthermore, since $m \geq 1$, we see that within U_1 , B_m is defined by the equation $|s_3/s_1| = 1$.

It is easy to see that within U_1 , U_2 is defined by $|s_2/s_1| = 1$.

Similarly, U_2 is defined by the equations

$$|s_1/s_2| \leq 1 \text{ and } |s_3/s_1| \leq 1,$$

and within U_2 , B_m is defined by $|s_3/s_2| = 1$ and U_1 by $|s_1/s_2| = 1$.

Pulling back via π_{HT} , we may view the s_i elements of $H^0(X_{\infty K^p, \omega^2})$. Now, since $V_1 := \pi_{\text{HT}}^{-1}(U_1)$, $V_2 := \pi_{\text{HT}}^{-1}(U_2)$, and $V_3 := \pi_{\text{HT}}^{-1}(B_m)$ are all affinoid perfectoid and the limit of affinoids at finite level, as in [24, proof of Theorem IV.1.1], we can find K_p such that:

- V_1 , V_2 , and V_3 are each the preimages of open affinoids V'_i in $X_{K_p K^p, \mathbb{C}_p}$.
- There exist sections

$$s_i^{(j)} \in H^0(V'_i, \omega^2)$$

for $i, j \in \{1, 2, 3\}$ and

$$\tilde{f}' \in H^0(V'_3, \mathcal{O})$$

such that

- after pullback to infinite level, for each i, j ,

$$s_i^{(j)}/s_i \in 1 + p^n H^0(V_j, \mathcal{O}^+),$$

- and, after pullback to infinite level,

$$\tilde{f}'/\tilde{f} \in 1 + p^n H^0(V_3, \mathcal{O}^+).$$

As in [24, proof of Theorem IV.1.1], this is enough to apply [24, Lemma II.1.1] to deduce the existence of a projective formal model \mathfrak{X} for $X_{K_p K^p, \mathbb{C}_p}$ equipped with an ample line bundle \mathfrak{L} which is an integral model for ω^2 , with affine opens \mathfrak{V}'_i which are formal models for V'_i , and such that $s_i^{(j)}$ comes from a section of \mathfrak{L} on \mathfrak{V}'_j and \tilde{f}' comes from a function on \mathfrak{V}'_3 . For each i , the sections

$$s_i^{(j)} \pmod{p^n}$$

glue to a global section \tilde{s}_i of \mathfrak{L}/p^n on \mathfrak{X} .

Now, because \tilde{s}_3 is nilpotent on $\mathfrak{V}'_1 - \mathfrak{V}'_3$ and $\mathfrak{V}'_2 - \mathfrak{V}'_3$, we find that for k sufficiently large,

$$\tilde{s}_3^k \tilde{f} \in H^0(\mathfrak{X}, \mathfrak{L}^k/p^n).$$

Furthermore, because \mathfrak{L} is ample, by possibly taking k larger, we may lift $\tilde{s}_3^k \tilde{f}'$ to

$$\omega_f \in H^0(\mathfrak{X}, \mathfrak{L}^k).$$

We claim that ω_f , viewed as an element of

$$H^0(\mathfrak{X}, \mathfrak{L}^k)[1/p] = H^0(X_{K_p K^p, \mathbb{C}_p}, \omega^{2k}),$$

evaluates at x to a function congruent to $f \bmod p^n$. Indeed,

$$\omega_f|_{X_{\infty K^p, x}} = (\omega_f|_{V_3})|_{X_{\infty K^p, x}} = (\tilde{f}'(XY)^k)|_{X_{\infty K^p, x}} = \tilde{f}'|_{X_{\infty K^p, x}} v^{2k}$$

since we chose v such that $v^2 = (XY)|_x$. Dividing by v^{2k} , we conclude, as \tilde{f}' is congruent to $\tilde{f} \bmod p^n$, and thus to $f \bmod p^n$ after restriction to $X_{\infty K^p, x}$.

□

6.5 Isomorphisms of Hecke algebras

In this section we prove Theorem A. In fact, we prove a slightly more general statement (cf. Remark 1.2.3):

Theorem 6.5.1. $\mathbb{T}'_{D^\times} \cong \mathbb{T}'_{\mathrm{GL}_2} \cong \mathbb{T}'_{\mathrm{mf}}$ as topological \mathbb{T}' algebras.

Proof. We consider the completed Hecke algebra $\mathbb{T}'_{\mathrm{aux}}$ corresponding to the collection

$$M_{k, K_p} := H^0(X_{K_p K^p}, \omega^k \otimes \det^{-1}) \otimes \mathbb{C}_p$$

for $k \geq 2$. Because the evaluation maps eval'_{x, K^p} when restricted to finite level are injective (cf. (6.3.0.2)), and by Lemma 6.4.1, their image is dense, we may apply Lemma 6.1.8 to conclude that $\mathbb{T}'_{\mathrm{aux}}$ is equal to the completed Hecke algebra of

$$\mathrm{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \mathbb{C}_p).$$

By Lemma 6.1.7, this is equal to \mathbb{T}'_{D^\times} (which is defined using \mathbb{Q}_p coefficients instead of \mathbb{C}_p coefficients).

Now, using the Eichler-Shimura isomorphism (via fixing $\mathbb{C}_p \cong \mathbb{C}$), we find Hecke equivariant maps

$$M_{k, K_p} \hookrightarrow H^1(Y_{K_p K^p}(\mathbb{C}), \mathrm{Sym}^{k-2} \mathbb{Q}^2 \otimes \mathbb{C}_p)$$

which induce isomorphisms on the image of \mathbb{T}' in the respective rings of endomorphisms. We deduce that \mathbb{T}'_{aux} is isomorphic to the completed Hecke algebra for the collection

$$H^1(Y_{K_p K^p}(\mathbb{C}), \text{Sym}^{k-2} \mathbb{Q}^2) \otimes \mathbb{C}_p$$

and, by Lemma 6.1.7, to the completed Hecke algebra for

$$H^1(Y_{K_p K^p}(\mathbb{C}), \text{Sym}^{k-2} \mathbb{Q}_p^2).$$

We may tensor each of these with the finite dimensional vector space $(\text{Sym}^{k-2} \mathbb{Q}_p^2)^*$ with trivial Hecke action without changing the completed Hecke algebra. Now, we have Hecke-equivariant injections

$$H^1(X_{K_p K^p}(\mathbb{C}), \text{Sym}^{k-2} \mathbb{Q}_p^2) \otimes (\text{Sym}^{k-2} \mathbb{Q}_p^2)^* \hookrightarrow \widehat{H}^1(Y)$$

describing (a subset of) the locally algebraic vectors in \widehat{H}^1 . The image is dense: in fact, it is dense already if we only consider $k = 2$ and varying K_p . Thus,

$$\mathbb{T}'_{\text{aux}} \cong \mathbb{T}'_{\text{GL}_2},$$

and we deduce

$$\mathbb{T}'_{D^\times} \cong \mathbb{T}'_{\text{GL}_2}.$$

We now show $\mathbb{T}'_{\text{GL}_2} \cong \mathbb{T}'_{\text{mf}}$, cf. [10, Remarks 5.4.2 and 5.4.3]. Arguing similarly and using the density of $GL_2(\mathbb{Z}_p)$ -algebraic vectors in \widehat{H}^1 (specifically of the ones of the form $(\text{Sym}^{k-2} \mathbb{Q}_p^2)^*$ as k varies; we do not need to also allow for arbitrary twists by a determinant), we conclude that $\mathbb{T}'_{\text{GL}_2}$ is equal to the completed Hecke algebra of the family

$$(M_{k, GL_2(\mathbb{Z}_p)})_k.$$

By Lemma 6.1.7 we can replace these with modular forms with \mathbb{Q}_p coefficients. Each of these spaces then admits a Hecke equivariant injection into the space $\mathbb{V}_{\text{Katz}} \otimes \mathbb{Q}_p$ of Katz p -adic modular forms, and the image is dense. Thus, by Lemma 6.1.8, $\mathbb{T}'_{\text{GL}_2} \cong \mathbb{T}'_{\text{mf}}$, and we conclude. □

CHAPTER 7

OVERCONVERGENT MODULAR FORMS

In this section we give a simple construction of overconvergent modular forms by working at infinite level. Using this construction, we extend the evaluation maps of Section 5 to overconvergent modular forms in order to prove Theorem B.

7.1 Overconvergent modular forms at infinite level

7.1.1 Reduction of structure group on \mathbb{P}^1

Let $\infty = [1 : 0]$ and consider the coordinate $z = Y/X$ for X, Y the standard sections of $O(1)$. For $\epsilon \in p^{\mathbb{Z}}$ we denote

$$B_\epsilon(\infty) := \{|z| \leq \epsilon\} \subset \mathbb{P}_{\mathbb{Q}_p}^1.$$

For $\epsilon \in p^{-\mathbb{N}}$ we denote by $\mathbb{Z}_p^{\times, \epsilon}$ the affinoid group which is an ϵ -neighborhood of \mathbb{Z}_p^\times inside \mathbb{G}_m . If $\epsilon = p^{-n}$ and we fix coset representatives a_i for $\mathbb{Z}/p^n\mathbb{Z}$, then

$$\mathbb{Z}_p^{\times, \epsilon} = \bigsqcup_i B_\epsilon(a_i) \subset \mathbb{A}^1.$$

Over $B_\epsilon(\infty)$, we consider the $\mathbb{Z}_p^{\times, \epsilon}$ -torsor

$$\mathcal{T}_{\mathbb{P}^1, \epsilon} : \mathbb{Z}_p^{\times, \epsilon} \times B_\epsilon(0) \rightarrow B_\epsilon(\infty) \tag{7.1.1.1}$$

which lies inside the canonical \mathbb{G}_m -torsor of bases for $O(1)$,

$$\mathbb{A}^2 - \{0\} \rightarrow \mathbb{P}^1. \tag{7.1.1.2}$$

The action of $\mathbb{Z}_p^{\times, \epsilon}$ and \mathbb{G}_m is by z^{-1} in both cases.

For $\epsilon = p^{-n}$, the natural GL_2 action on (7.1.1.2) restricts to a $\Gamma_0(p^n)$ -action on $\mathcal{T}_{\mathbb{P}^1, \epsilon}$.

We consider the tower $(\mathcal{T}_{\mathbb{P}^1, \epsilon})_{K^p}$ with the natural action of $\Gamma_0(p^n) \times GL_2(\mathbb{A}_f^{(p)})$ twisted by $\det_{\text{ur}} \cdot \det_p$ (which takes values in \mathbb{Z}_p^\times).

7.1.2 Pulling back to X_∞

We let

$$U_{\epsilon K^p} := \pi_{\text{HT}}^{-1}(B_\epsilon(\infty)).$$

and

$$\mathcal{T}_{\infty, \epsilon, K^p} := \pi_{\text{HT}}^* \mathcal{T}_{\mathbb{P}^1, \epsilon}.$$

Remark 7.1.3. The simplest way to interpret the pullback is geometrically by taking the fiber product over π_{HT} with the geometric torsor and then forming the sheaf of sections. We can also describe the sheaf of sections of $\mathcal{T}_{\infty, \epsilon, K^p}$ explicitly: it is the subsheaf of $\omega|_{U_{\epsilon K^p}}$ consisting of non-vanishing sections s such that $(X/s, Y/s)$ lies in $\mathbb{Z}_p^{\times, \epsilon}(\mathcal{O}) \times B_\epsilon(\mathcal{O})$.

7.1.4 Modular forms of weight κ

Let E/\mathbb{Q}_p be a complete extension and let $\kappa : \mathbb{Z}_p^\times \rightarrow E$ be a continuous character which extends to $\mathbb{Z}_p^{\times, \epsilon}$ (note, any continuous character of \mathbb{Z}_p^\times is locally analytic, and thus extends for some ϵ).

Definition 7.1.5. The infinite level sheaf of weight κ modular forms is the $\Gamma_0(p^n) \times GL_2(\mathbb{A}_f^{(p)})$ -equivariant sheaf on $X_{\infty K^p, E}$

$$\omega^\kappa := (\mathcal{T}_{\infty, \epsilon, K^p} \times^\kappa \mathcal{O})_{K^p}.$$

Remark 7.1.6. We obtain the same sheaf if we first take the pushout

$$\mathcal{O}(\kappa) \otimes \kappa \circ \chi_{\text{HT}} := \mathcal{T}_{\mathbb{P}^1, \epsilon} \times^\kappa \mathcal{O}$$

and then pull back via π_{HT} .

7.2 Smooth vectors

On $X_{\infty K^p}$, we denote by \mathcal{O}^{sm} the sheaf of smooth vectors for the $GL_2(\mathbb{Q}_p)$ action, i.e., on a quasi-compact $U \subset X_{\infty K^p}$,

$$\mathcal{O}^{\text{sm}}(U) = \bigcup_{K_p} \mathcal{O}(U)^{K_p}$$

where the union is over all sufficiently small compact opens $K_p \subset GL_2(\mathbb{Q}_p)$. This definition makes sense, since any quasi-compact is preserved by a compact open (the action is continuous in the sense of [22]).

Remark 7.2.1. Alternatively, over $Y_{\infty K^p}$, if we fix a compact open K_p , then \mathcal{O}^{sm} is the restriction of the structure sheaf \mathcal{O} on the pro-étale site of the finite level modular curve, $Y_{K_p K^p, \text{proét}}$ to $Y_{\infty K^p}$, viewed as an object of the pro-étale site (whereas the structure sheaf we have been considering on $Y_{\infty K^p}$ would be the restriction of $\widehat{\mathcal{O}}$).

Similarly, for any $U \subset X_{\infty K^p}$ preserved by an open $G_0 \subset GL_2(\mathbb{Q}_p)$ and a G_0 -equivariant sheaf on U , \mathcal{F} , it makes sense to form \mathcal{F}^{sm} , the sheaf of smooth sections of \mathcal{F} .

Lemma 7.2.2. *On any rational sub-domain of U_ϵ , $\mathcal{O}^{\text{sm}}(U)$ is dense in $\mathcal{O}(U)$.*

Proof. This follows from Theorem 3.2.11, since functions pulled back from finite level are smooth. □

Lemma 7.2.3. *There is a covering of U_ϵ by rational subsets V such that $\mathcal{T}_\epsilon^{\text{sm}}(V) \neq \emptyset$.*

Proof. By Theorem 3.2.11, U_ϵ is the pre-image of an open affinoid $U_{\epsilon K^p}$. We may cover $U_{\epsilon K^p}$ by rational sub-domains where ω is trivialized. Pulling back to infinite level gives a cover by rational sub-domains V of U_ϵ where $\omega|_V$ admits a smooth non-vanishing section s . Then, because $\mathcal{O}^{\text{sm}}(V)$ is dense in $\mathcal{O}(V)$, and $\mathcal{T}_\epsilon(V) \cdot \frac{1}{s}$ is open in $\mathcal{O}(V)$ (it is an orbit of

$\mathbb{Z}_p^{\times, \epsilon}(\mathcal{O}(V))$, we find that there exists

$$f \in \mathcal{O}^{\text{sm}}(V) \cap \left(\mathcal{T}_\epsilon(V) \cdot \frac{1}{s} \right).$$

Then, $f \cdot s$ is in $\mathcal{T}_\epsilon^{\text{sm}}(V)$. □

From Lemma 7.2.3, we conclude $\mathcal{T}_\epsilon^{\text{sm}}$ is a $\mathbb{Z}_p^{\times, \epsilon}(\mathcal{O}^{\text{sm}})$ -torsor on U_ϵ . Thus we obtain

Theorem 7.2.4. $(\omega^\kappa)^{\text{sm}}$ is locally free of rank 1 over \mathcal{O}^{sm} .

Proof. We take a rational covering of U_ϵ by V as in Lemma 7.2.3. If $s \in \mathcal{T}_\epsilon^{\text{sm}}(V)$ then

$$(s, 1) \in \omega^\kappa(V) = \mathcal{T}_\epsilon \times^\kappa \mathcal{O}(V)$$

is a basis for $\omega^\kappa(V)$, and an element of $(\omega^\kappa)^{\text{sm}}(V)$. For any $s' \in (\omega^\kappa)^{\text{sm}}(V)$

$$s'/s \in \mathcal{O}^{\text{sm}}(V)$$

as it is fixed by the intersection of the open stabilizers of s' and s . Thus we conclude. □

Remark 7.2.5. We could also construct $(\omega^\kappa)^{\text{sm}}$ as $\mathcal{T}_\epsilon^{\text{sm}} \times^\kappa \mathcal{O}^{\text{sm}}$. We could *not*, however, take smooth vectors first on \mathbb{P}^1 and then pullback – on \mathbb{P}^1 there are no smooth vectors!

Definition 7.2.6. For $w \leq \epsilon$,

$$M_{\kappa, K^p}^w := H^0(\pi_{\text{HT}}^{-1} B_w(\infty), \omega^{\kappa, \text{sm}}),$$

and

$$M_{\kappa, K^p}^\dagger := \varinjlim_w M_{\kappa, K^p}^w.$$

Remark 7.2.7. The space M_κ^\dagger should be thought of as containing information about over-convergent modular forms of weight κ at *all* finite levels. We highlight, however, that rather

than working with some fixed finite level, in our setup it is simplest to work with all finite levels at once by considering the smooth vectors. Furthermore, for our application to Jacquet-Langlands, this representation theoretic characterization is the one we are most interested in!

7.3 Finite level

In this section we refine the construction of 7.2 to construct overconvergent modular sheaves at finite level and compare with the construction of Pilloni [21].

We fix a compact open $K_p \subset GL_2(\mathbb{Q}_p)$ and denote by

$$\pi_{K_p} : X_{\infty K^p} \rightarrow X_{K_p}$$

the natural map.

Lemma 7.3.1.

$$|X_{K_p K^p}| = |X_{\infty K^p}|/K_p$$

and for $W \subset X_{K_p K^p}$ open,

$$\mathcal{O}(\pi_{\text{HT}}^{-1}(W))^{K_p} = \mathcal{O}(W)$$

Proof. Over the open modular curve $Y_{K_p K^p}$, $Y_{\infty K^p}$ is a profinite étale cover with structure group K_p , and we find $|Y_{\infty K^p}|/K_p = |Y_{K_p K^p}|$. Moreover, from the sheaf property for $\widehat{\mathcal{O}}$ on $Y_{K_p K^p, \text{proét}}$ and

$$\widehat{\mathcal{O}}|_{Y_{K_p K^p, \text{an}}} = \mathcal{O}_{Y_{K_p K^p}}, \quad \widehat{\mathcal{O}}|_{Y_{\infty K^p, \text{an}}} = \mathcal{O}_{Y_{\infty K^p}},$$

(cf [23]), we conclude that for $W \subset Y_{K_p K^p}$ open,

$$\mathcal{O}(\pi_{\text{HT}}^{-1}(W))^{K_p} = \mathcal{O}(W).$$

Thus it remains only to extend these results to the boundary. Using the sheaf property, we

see that it suffices to find a basis of neighborhoods B of the boundary in $X_{K_p K^p}$ such that $|B| = |\pi_{\text{HT}}^{-1}(B)|/K_p$ and $\mathcal{O}(\pi_{\text{HT}}^{-1}(B))^{K_p} = \mathcal{O}(B)$. Such a basis is given by taking arbitrarily small q -balls around the cusps, where the computations can be made completely explicit. \square

Corollary 7.3.2. *If \mathcal{V} is a vector bundle on $V \subset X_{K_p K^p}$ and $W \subset V$ is an open subset,*

$$\mathcal{V}(W) = \left(\pi_{\text{HT}}^*(\mathcal{V})(\pi_{K_p}^{-1}(W)) \right)^{K_p}.$$

Theorem 7.3.3. *For K_p preserving U_ϵ , let $U_{\epsilon, K_p} = U_\epsilon/K_p$. There is a line bundle $\omega_{K_p}^\kappa$ on U_{ϵ, K_p} such that*

$$\omega^\kappa = \pi_{K_p}^* \omega_{\Gamma_0(p^n)}^\kappa.$$

Proof. It will suffice to show that \mathcal{T}_ϵ is the pullback of a $\mathbb{Z}_p^\times(\mathcal{O})$ -torsor $\mathcal{T}_{K_p, \epsilon}$ on $U_{K_p, \text{ét}}$ via π_{K_p} . Indeed, then we may push-out by κ at finite level to obtain an étale line bundle $\omega_{K_p}^\kappa$, which is automatically a line bundle on the analytic site.

Combining Lemma 7.3.1 and Corollary 7.3.2, we find that $(\pi_{K_p, *}\mathcal{T}_\epsilon)^{K_p}$ is such a torsor, and we conclude. \square

Remark 7.3.4. In fact, U_{ϵ, K_p} is affinoid because for some $K'_p \subset K_p$, U_{ϵ, K'_p} is affinoid by 3.2.11, and quotients of affinoids by finite groups are again affinoid by [13].

Corollary 7.3.5. *For $K_p \subset \Gamma_0(p^n)$ and $W \subset U_{\epsilon, K_p}$,*

$$\omega^\kappa(\pi_{K_p}^{-1}(W))^{K_p} = H^0(W, \omega_{K_p}^\kappa).$$

7.3.6 Compactness of U_p

For $w \leq \epsilon$

$$g_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \tag{7.3.6.1}$$

induces a map

$$B_w(\infty) \rightarrow B_{w/p}(\infty)$$

which lifts to a map

$$\mathcal{T}_{\mathbb{P}^1, \epsilon}|_{B_w(\infty)} \rightarrow \mathcal{T}_{\mathbb{P}^1, \epsilon}|_{B_{w/p}(\infty)}.$$

For a finite level K_p , the Hecke operator U_p on $H^0(U_{w/p}, \omega^\kappa)_p^K$ induced by g_p^{-1} factors as

$$\begin{aligned} H^0(U_{w/p}, \omega^\kappa)^{K_p} &\xrightarrow{(g_p \cdot)^*} H^0(U_w, \omega^\kappa)^{g_p \Gamma_0(p^n) g_p^{-1}} \xrightarrow{\text{res}} H^0(U_{w/p}, \omega^\kappa)^{g_p K_p g_p^{-1}} \\ &\rightarrow H^0(U_{w/p}, \omega^\kappa)^{g_p K_p g_p^{-1} \cap K_p} \xrightarrow{\text{trace}} H^0(U_{w/p}, \omega^\kappa)^{K_p}. \end{aligned}$$

Because $\overline{U_{w/p}} \subset U_w$, we find $\overline{U_{w/p, K_p}} \subset U_{w, K_p}$, and interpreting the restriction map res at finite level we deduce it is a compact map, and thus the Hecke operator U_p is also compact.

7.3.7 Comparison with Pilloni

In [21], Pilloni gives a closely related construction of overconvergent modular forms by working at finite level. Over neighborhoods of the ordinary locus, he constructs open subsets \mathcal{F} of the torsor of bases \mathcal{T} for ω , and then considers functions on \mathcal{F} that transform under a character of \mathbb{Z}_p^\times . It can be verified that, for suitable choices, our torsor \mathcal{T}_ϵ is contained in Pilloni's \mathcal{F} , and that his construction agrees with taking the push-out. Alternatively, one can argue as in [21] that the Eisenstein family is overconvergent in our sense in order to compare with Coleman's construction, and thus indirectly with Pilloni's.

7.4 The evaluation maps

We assume $\check{F} \subset E$. We fix a weight character $\kappa : \mathbb{Z}_p^\times \rightarrow E^\times$, an $\epsilon = p^{-n}$ such that κ extends to $\mathbb{Z}_p^{\times, \epsilon}$ and $w \leq \epsilon$ as before. We also fix CM data as in 5.1 such that the corresponding

point $x \in \mathbb{P}^1(F)$ lies in $B_w(\infty)$. We note that taking points closer to ∞ corresponds to taking smaller orders $R \subset F$ (we need that R is contained in an ϵ -neighborhood of \mathbb{Z}_p). For our choice, we have $j_M(R^\times) \subset \Gamma_0(p^n)$.

Remark 7.4.1. Because in our set-up we fix a radius of convergence, it may seem as though we have lost information about forms which don't overconverge to this fixed radius. However, because we allow arbitrary level, any form can be extended using the contracting operator

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

at the price of increasing the level. This is related to the standard trick for extending finite slope forms at level $\Gamma_1(p^m)$.

We now proceed as in Section 5. For κ_1, κ_2 characters of $\mathbb{Z}_p^{\times, \epsilon}$ with values in E , we denote $\tau_{\kappa_1, \kappa_2}$ the character $a \mapsto \kappa_1(a)\kappa_2(\bar{a})$ of R^\times (note that $R^\times \subset \mathbb{Z}_p^{\times, \epsilon}(F)$, so that it makes sense to evaluate κ). The sheaf $\mathcal{O}(\kappa) \cdot (\det_p \det_{\text{ur}})^\kappa \det_{\text{ur}}$ (cf. Remark 7.1.6) has a natural $R^\times \times GL_2(\mathbb{A}_f)$ -equivariant trivialization after restriction to x , and pulling this back via π_{HT} we obtain an analog of (5.7.0.1):

$$\tau_{0, \kappa} \cdot (\kappa \circ \det_{\text{ur}}) \cdot (\det_{\text{ur}})^{-1} \mathcal{O} \xrightarrow{\sim} \omega^\kappa \otimes \det_{\text{dR}} \quad (7.4.1.1)$$

where here the isomorphism is as $R^\times \times GL_2(\mathbb{A}_f)$ -equivariant bundles on $(X_{x, K^p, E})_{K^p}$.

Remark 7.4.2. Because we have restricted to R^\times , no p will appear in \det_{ur} , and thus it takes values in \mathbb{Z}_p^\times as is necessary to compose with κ .

Evaluating this trivialization on sections of $\omega^{\kappa, \text{sm}}$ we obtain maps

$$M_{\kappa, K^p}^w \rightarrow \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, \widehat{E \cdot F^{\text{ab}}}).$$

Because the elements of M_{κ, K^p} are smooth, the action of some sufficiently small open subset

of R^\times on the image is via the character $\tau_{0,\kappa-1}$. By the reciprocity law (Lemma 5.3.1), the functions in the image take values in the $\tau_{0,\kappa-1}$ -isotypic component for a sufficiently small open of $\text{Gal}(E \cdot F^{\text{ab}}/E)$, and thus, after dividing by a period, in $E \cdot F^{\text{ab}}$. Finally, using the trivialization (5.5.0.1) of \det_{ur} over $D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K^p$, we obtain Hecke-equivariant evaluation maps

$$\text{eval}_{x,K^p}^{\kappa,w} : M_{\kappa,K^p}^w \rightarrow \text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K^p, E \cdot F^{\text{ab}}). \quad (7.4.2.1)$$

We have:

Theorem 7.4.3. *The maps $\text{eval}_{x,K^p}^{\kappa,w}$ are Hecke-equivariant injections, and factor through the $[\text{lie}\kappa - 1, -1]$ -isotypic component for the $\text{Lie}_{\mathbb{Q}_p} F^\times$ -action on the F^\times -analytic vectors.*

Proof. In the construction we have seen everything except that the maps are injections. The argument for injectivity is essentially the same as the classical case: by Corollary 7.3.5, a K_p -invariant section of ω^κ over U_w is the same as a finite level section in $H^0(U_w, K^p, \omega_{K^p}^\kappa)$. The image of $D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K^p$ in this one-dimensional quasi-compact space is infinite in each component, and thus a section which vanishes along it must be zero. \square

CHAPTER 8

LOCAL-GLOBAL COMPATIBILITY

Both Knight [19] and Scholze [22] have constructed *local* p -adic Jacquet-Langlands correspondences, and a proof that the correspondences agree has been announced by Chojecki-Knight [7]. For E/\mathbb{Q}_p a finite extension and Π a continuous unitary admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on an E -Banach space, we denote by $J(\Pi)$ the continuous unitary $D^\times(\mathbb{Q}_p)$ representation associated to Π under this correspondence.

In [19] and [22], the correspondence J is shown to satisfy local-global compatibility with the completed cohomology of Shimura curves. Below we make a (weak) local-global compatibility conjecture for the space of naive automorphic forms on the definite quaternion algebra D^\times , which can also be thought of as a completed H^0 for the corresponding zero-dimensional Shimura variety.

Fix a $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ and let $\mathbb{T}' \subset \mathbb{T}_{\mathrm{abs}}$ be an unramified Hecke algebra for level K^p (i.e., \mathbb{T}' is generated by the Hecke algebra at l for all but finitely many l where K^p factors as $K^{p,l} \cdot K_l$ for K_l maximal compact). As in Chapter 6, we denote by $\mathbb{T}'_{\mathrm{mf}}$ the corresponding completed Hecke algebra for Katz p -adic modular functions. If E/\mathbb{Q}_p is a finite extension and $\lambda : \mathbb{T}'_{\mathrm{mf}} \rightarrow E$ is a character, then as in [14, Theorem II] we obtain a semi-simple representation ρ_λ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a two-dimensional E -vector space (after possibly enlarging E). By Theorem 6.5.1, the completed Hecke algebra \mathbb{T}'_{D^\times} for

$$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p$$

is equal to $\mathbb{T}'_{\mathrm{mf}}$, and thus we may also associate a Galois representation ρ_λ to a character λ of \mathbb{T}'_{D^\times} . Via the p -adic Langlands correspondence [8], we then obtain a $\mathrm{GL}_2(\mathbb{Q}_p)$ -representation

$$\Pi(\rho_\lambda|_{G_{\mathbb{Q}_p}}),$$

and finally a $D^\times(\mathbb{Q}_p)$ -representation

$$J(\Pi(\rho_\lambda|_{G_{\mathbb{Q}_p}})).$$

On the other hand, we also obtain a $D^\times(\mathbb{Q}_p)$ -representation from λ by considering the isotypic component

$$\text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, E)[\lambda],$$

i.e. the set of all vectors transforming under \mathbb{T}' via λ . Our conjecture relates these two representations:

Conjecture 8.0.1. *Let E/\mathbb{Q}_p be a finite extension, let \mathbb{T}'_{D^\times} as above, and let $\chi : \mathbb{T} \rightarrow E$ be a character such that ρ_χ is irreducible and defined over E . Then,*

$$\text{Cont}(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p, E)[\lambda]$$

is a finite direct sum of copies of $J(\Pi(\rho_\chi|_{G_{\mathbb{Q}_p}}))$, and can be made non-zero by increasing the ramified level without adding ramified primes.

Remark 8.0.2. One could make the conjecture more precise by including the local Langlands representations at ramified $l \neq p$.

Remark 8.0.3. There is an obvious strategy for attacking Conjecture 8.0.1: as in Remark 4.3.2, if we form the space

$$(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p)^{\text{ad}} \times \widehat{\text{LT}}_\infty$$

then, morally, the quotient by the diagonal action of $D^\times(\mathbb{Q}_p)$ is the infinite level super-singular locus. Functions here can then be related to the completed cohomology of the modular curve which is known to realize the p -adic local Langlands correspondence in most

cases. On the other hand, the quotient by the $GL_2(\mathbb{Q}_p)$ action is morally

$$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) / K^p \times \mathbb{P}^1$$

with the diagonal $D^\times(\mathbb{Q}_p)$ -action, which is naturally related to Scholze's construction of the p -adic Jacquet-Langlands [22]. There are technical obstacles to making this precise.

Under some mild assumptions on ρ_λ , by [19, Theorem 1.0.4], the locally algebraic vectors in $J(\Pi(\rho_\lambda|_{G_{\mathbb{Q}_p}}))$ are “what you would expect” – i.e., zero except in the situation where the smooth Jacquet-Langlands correspondence applies, in which case they are equal to the corresponding smooth representation tensored with an algebraic representation determined by the Hodge-Tate weights. In particular, the locally algebraic vectors are finite dimensional and thus a closed subspace, and it is natural to conjecture (cf. [19]) that they are never dense, i.e. that $J(\Pi(\rho_\lambda|_{G_{\mathbb{Q}_p}}))$ does not consist only of locally-algebraic vectors.

Corollary 8.0.4. *If Conjecture 8.0.1 holds for λ coming from an overconvergent modular form as in Corollary C and $\rho_\lambda|_{G_{\mathbb{Q}_p}}$ satisfies the hypotheses of [19, Theorem 1.0.4], then the locally algebraic vectors in $J(\Pi(\rho_\lambda|_{G_{\mathbb{Q}_p}}))$ are not dense.*

Proof. If the locally algebraic vectors are empty, then this follows immediately from Corollary C, which produces a non-zero vector. Otherwise, the modular form giving rise to λ is classical of weight $k \geq 2$. Applying [19, Theorem 1.0.4], we find that the locally algebraic vectors of

$$J(\Pi(\rho_\lambda|_{G_{\mathbb{Q}_p}})),$$

restricted to a sufficiently small open subgroup of D^\times , are isomorphic to a twist of $\text{Sym}^{k-2} E^2$. The Lie algebra weights of a maximal torus in the norm one elements of D^\times acting on these locally algebraic vectors are thus in $[-(k-2), k-2]$.

On the other hand, by Corollary C, there exists a non-zero vector of weight k , which thus is not locally algebraic. □

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