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INSCRIPTION AND p-ADIC PERIODS

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ABSTRACT. We introduce the category of inscribed v-sheaves as a differential extension of the theory of diamonds and v-sheaves, then apply it to the study of p-adic periods. We upgrade natural period domains in p-adic Hodge theory to inscribed v-sheaves, including Schubert cells in B^+_{dR}-affine Grassmannians, their generalized Newton strata, and the moduli of mixed characteristic local shtuka with one leg; along the way we also establish new results on the bitorsor structure of moduli of modifications and a very general two towers isomorphism. We then compute the tangent bundles of these spaces and the derivatives of natural maps between them in terms of the fundamental exact sequences of p-adic Hodge theory. These computations encode interesting structures: for example, the derivative of the Bialynicki-Birula map enforces Griffiths transversality in p-adic Hodge theory, and the structure of the tangent and normal bundles of Newton strata enriches previous cohomological dimension computations. Miniscule moduli of mixed characteristic local shtukas with one leg are the same as infinite level local Shimura varieties, thus our results specialize to a computation of the tangent bundles of infinite level local Shimura varieties and of the derivatives of their Hodge and Hodge-Tate periods maps. Assuming a version of the Igusa stacks product conjecture that is known in many cases, we use similar methods to inscribe infinite level global Shimura varieties then compute their tangent bundles and the derivatives of their Hodge and Hodge-Tate period maps. We discuss consequences of these computations assuming some natural conjectures in "differential topology for diamonds," and, in particular, generalize a cohomological smoothness result of Ivanov and Weinstein for infinite level EL local Shimura varieties.

CONTENTS

1. Introduction

Over the past decade, the fundamental building blocks of p-adic geometry have shifted from those of rigid analytic geometry, Tate's convergent power series rings, to those of perfectoid geometry, Scholze's perfectoid algebras (characterized by the existence of approximate pth roots). This shift has increased the scope and power of the theory, even in the study of rigid analytic varieties and their cohomology. For example, by building from perfectoid algebras, one obtains new period maps in p-adic Hodge theory whose domain and/or codomain does not exist in the classical theory. However, this shift comes at a price: by moving away from the familiar convergent power series rings, we lose access to the differential toolkit of tangent spaces and derivatives that is so fundamental in classical geometric reasoning.

Example 1.0.1. Let $C/\mathbb{Q}_p$ be an algebraically closed non-archimedean extension. A basic example of a perfectoid C-algebra is the completion $C\langle t^{\pm 1/p^\infty}\rangle$ of $\bigcup_{n\geq 1} C[t^{\pm 1/p^n}]$ for the supremum norm on the coefficients. Geometrically, this is the ring of functions on the perfectoid annulus, a Galois $\mathbb{Z}_p(1)$-cover of the rigid analytic annulus $|t| = 1 \subseteq \mathbb{A}^1$ whose ring of functions is $C\langle t^{\pm 1}\rangle$. The module of continuous Kähler differentials of $C\langle t^{1/p^\infty}\rangle$ over C is trivial: any continuous derivation d with values in a Banach $C\langle t^{1/p^\infty}\rangle$-module is zero because of the formula $p^n d\log t^{1/p^n} = d\log t$.

This phenomenon is general: the existence of approximate p-power roots forces any continuous derivation on a perfectoid algebra to be identically zero. Thus, to obtain a broadly applicable differential theory for perfectoid spaces, one cannot proceed directly via the Kähler approach as in classical rigid analytic, complex analytic, or algebraic geometry. This issue propagates more broadly to the theory of *diamonds*, which are functors on perfectoid spaces constructed as quotients of representable functors by pro-étale equivalence relations, as well as the more general *v-sheaves* and *v-stacks* that arise naturally in moduli problems related to p-adic cohomology.

Nevertheless, it is clear that some diamonds and v-sheaves do have natural tangent bundles. For example, if $L/\mathbb{Q}_p$ is a non-archimedean extension, then smooth rigid analytic varieties over L embed, by their functor of points on perfectoid algebras over L, fully faithfully into the category of diamonds over Spd$L$, and certainly we know how to define the tangent bundle of a smooth rigid analytic variety. More recently, Fargues and Scholze [7], in the context of their Jacobian criterion for cohomological smoothness, have defined Banach-Colmez Tangent Bundles for moduli spaces of sections of smooth quasi-projective adic spaces over Fargues-Fontaine curves. Here, the Tangent Bundle is a Vector Bundle, i.e. a sheaf of modules for the topological constant sheaf $\underline{\mathbb{Q}_p}$ (here we follow the precepts of Colmez Capitalization in our nomenclature). The same diamond can sometimes be constructed in different ways as a moduli of sections, leading to distinct Tangent Bundles.

Example 1.0.2. Let C be an algebraically closed perfectoid field in characteristic p. Then, the moduli of sections of the vector bundle $\mathcal{O}(1/n)$ on the Fargues-Fontaine curve $X_C/\mathbb{Q}_p$ is represented by the open perfectoid unit disk over C. The Tangent Bundle assigned to it by the Fargues-Scholze construction is the constant Banach-Colmez space $\mathrm{BC}(\mathcal{O}(1/n))$. For varying n, these are distinct as Vector Bundles on the open perfectoid unit disk.

Thus the Tangent Bundle assigned by such a construction is not intrinsic to the diamond, and can be thought of as a type of additional differential structure akin to a differential manifold structure on a topological space. The first purpose of the present work is to introduce a notion of differential structure from which such a Tangent Bundle arises naturally. This is the theory of inscribed diamonds and v-sheaves, where the objects are not functors on perfectoid spaces but instead functors on certain finite locally free thickenings of Fargues-Fontaine curves[1]. These thickenings are a natural choice of test objects that play a role similar to the artinian rings used in the deformation theory of Galois representations — in fact, because of a condition we impose to ensure that basic strata in the inscribed moduli of G-bundles are open, the thickenings we consider can be thought of as "artinian" effective Banach-Colmez algebras over $\mathbb{Q}_p$ with residue field $\mathbb{Q}_p$.

In particular, our formalism allows us to extract Tangent Bundles as above from a simple construction of internal tangent bundles by Weil restriction. Indeed, any inscribed v-sheaf $\mathcal{S}$ admits an underlying v-sheaf $\mathcal{S}_0$ and a tangent bundle $T_{\mathcal{S}}$ obtained by "adding an ϵ". The tangent bundle $T_{\mathcal{S}}$ is itself an inscribed v-sheaf, and $(T_{\mathcal{S}})_0$ can be thought of as a Tangent Bundle to $\mathcal{S}_0$. As in the Grothendieck/Schlessinger deformation theory, to obtain the group structure on $T_{\mathcal{S}}$, we impose in the definition the condition that $\mathcal{S}$ should transform certain simple coproducts into products.

This theory encompasses naturally the tangent bundles of rigid analytic varieties and the Tangent Bundles of the Fargues-Scholze Jacobian criterion. In fact, both arise from moduli of sections constructions over different loci on the Fargues-Fontaine curve, and we also treat more general loci.

[1] The analogy here is with an inscription on a rock-theoretic diamond, which is a piece of extra identifying information laser-etched along the edge at its widest part.

Remark 1.0.3. There are other categories of thickenings one could use. One reason we use finite locally free thickenings is that they are the largest natural category for which we can easily establish a GAGA equivalence between algebraic and analytic thickenings. However, we have tried to set up the theory to be compatible with the eventual use of other categories of thickenings. In particular, we hope that many of our computations of tangent bundles and derivatives will extend to a category of thickenings as analytic stacks as in the theory of analytic prismatization that has been announced by Anschütz, le Bras, Rodriguez Camargo, and Scholze.

One goal of this theory is to provide a formalism for applying differential techniques to the study of p-adic period maps that arise in the study of p-adic cohomology. To that end, we give natural inscriptions on p-adic period domains: the B_{dR}^+-affine Grassmannians, their Schubert cells and generalized Newton strata, and the moduli of mixed characteristic local shtukas with one leg (in the minuscule case, these latter are also known as infinite level local Shimura varieties). The natural maps between these, including Hodge and Hodge-Tate period maps and their lattice refinements, are also upgraded to the inscribed setting, and we give explicit descriptions of the tangent bundles and the derivatives of these maps using the fundamental exact sequences of p-adic Hodge theory (see Theorems A, B, and C).

These p-adic period domains include all of the spaces that appear clasically as the *codomain* of a period map in p-adic Hodge theory. As for the domains of period maps, the moduli of mixed characteristic local shtukas with one leg can be used to understand the domains of period maps for all flat crystalline local systems, but do not include, e.g., infinite level Shimura varieties or the more general infinite level domains of period maps arising from de Rham local systems over smooth rigid varieties. However, inspired by a construction announced by Scholze in the context of analytic prismatization, we also construct inscribed global Shimura varieties assuming a version of the Igusa stacks fiber product conjecture and then adapt the methods used in the local case to compute their tangent bundles and differentiate their period maps (see Theorem D). The Igusa stacks fiber product conjecture in this form is known for many PEL Shimura varieties [29] and in the case of compact Hodge type Shimura varieties [5], so that our construction and computations are unconditional in these cases.

We note that, in the case of moduli of mixed characteristic local shtuka, we also establish a general two towers isomorphism (generalizing the now classical isomorphism between the infinite level Lubin-Tate and Drinfeld spaces) in the inscribed setting that also applies beyond the basic case where it has previously been studied (see Section 9.3). One of the key tools in all of our computations in this local setting is the use of a simple bitorsor structure on the unbounded moduli space, which appears to have been previously unexploited in the literature; a related torsor structure for the unbounded Beauville-Laszlo map also plays an important role in our constructions and computations for global Shimura varieties. The final key tool that allows us to related the inscribed infinite level local and global Shimura varieties to inscribed finite level local and global Shimura varieties is a novel study of Hodge period maps in the inscribed setting.

We highlight two consequences of the computations in the present work: First, the derivative of the Bialynicki-Birula map from a Schubert cell in the B^+_{dR}-affine Grassmannian to the corresponding flag variety gives a natural conceptual explanation of Griffiths transversality (see Theorem A and Remark 2.2.2).

Second, one expects that there should be a theory of *differential topology for diamonds*, whereby a differential structure is used to establish "topological" results about the underlying the v-sheaf. The Fargues-Scholze Jacobian criterion for cohomological smoothness is the seminal such result, and we expect that a version of it should apply to the more general tangent bundles that we construct here. We study the implications of this expectation in detail for moduli of mixed characteristic local shtukas with one leg. In particular, in Theorem E we prove a result that, given such a generalization, would imply a simple description of the cohomologically smooth locus. In the minuscule EL case we obtain this description unconditionally (Corollary F) by showing our construction agrees with a moduli of sections construction to which the Fargues-Scholze criterion does apply, generalizing the *basic* minuscule EL case due to Ivanov and Weinstein [14]. The ℓ-adic cohomology of these moduli spaces plays a central role in the local Langlands correspondence, and this description of the cohomologically smooth locus is a finiteness result that gives a qualitative interpretation of a relation between representation-theoretic constructions, formal models, and ℓ-adic cohomology that was first observed in work of Weinstein [28] for the height two Lubin-Tate tower.

Continuining in the vein of differential topology for diamonds, the structure of the tangent bundle often furnishes a natural prediction for the (pre)perfectoid locus of a diamond. In Section 2.5 we discuss this briefly in the context of quotients of infinite level local and global Shimura varieties, but defer a more systematic discussion to a sequel [10] where we give a functorial construction of Tangent Bundles for general p-adic manifold fibrations over smooth rigid analytic varieties. That construction is based on a connection between the Tangent Bundles arising from inscribed structures that we construct here in the case of local and global Shimura varieties and the geometric Sen morphism of Pan [20] and Camargo [2]; of the many interesting aspects of this connection, let us highlight here only that there is a natural space of locally analytic functions on a p-adic manifold fibration which admits a derivation action by an $\mathcal{O}$-linearized version of our Tangent Bundles and that this derivation action is intimately related to the key annihilation property of the geometric Sen morphism.

Although connected at a conceptual level, the results of [10] are in large part logically independent from the present work. However, we expect that the Tangent Bundles of general p-adic manifold fibrations studied in [10] should also arise from a natural inscribed structure. In a separate sequel [11], we establish this in by constructing p-adic twistors attached to many $\mathbb{Q}_p$-local systems (where already our definition of p-adic twistors uses the inscribed formalism). In particular, the results of [11] will recover from a different perspective the inscribed structures on local and global Shimura varieties that are constructed in the present work (but not the more general non-minuscule constructions given here). In fact, the construction of twistors in [11] uses results on the inscribed structures on period domains contained in the present work and is a natural outgrowth of the study of the inscribed Hodge period map that is used here to compare finite level and infinite level inscribed structures for local and global Shimura varieties.

In Section 2, we give precise statements of our main results. The organization of the remainder of the article can be found in Section 2.7.

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2. Statements of main results

2.1. **Inscribed v-sheaves.** We write Perf[2] for the category of perfectoid spaces in characteristic p. We will employ its v-topology and its étale topology. We fix a finite extension $E/\mathbb{Q}_p$ and, for $P \in \mathrm{Perf}/\mathrm{Spd}\mathbb{F}_q$, we write $X_{E,P}$ for the associated Fargues-Fontaine curve over E, an adic space over $\mathrm{Spa}E$. A key role below will be played by the topologically constant v-sheaf of rings $\underline{E}$ on Perf and its defining formula in terms of $X_{E,P}$:

$$\underline{E}(P) = \mathrm{Cont}(|P|, E) = H^0(X_P, \mathcal{O}_{X_P}). \tag{2.1.0.1}$$

Previous constructions of Banach-Colmez Tangent Bundles [7, 14] via a heuristic implicit in the Fargues-Scholze Jacobian criterion have treated the Tangent Bundle of a v-sheaf S as an E-Vector Bundle over S, i.e. a sheaf of $\underline{E}$-modules over S. As explained in the discussion surrounding Example 1.0.2, it is misleading to refer to such an E-Vector Bundle as *the* Tangent Bundle of S, since it cannot typically be extracted from S itself but instead depends on the specific construction of S as a moduli of sections. To address this issue, we work with functors on a category $X_{E,\square}^{\mathrm{lf}^+}$ whose objects are the finite locally free thickenings of $\mathcal{X}/X_{E,P}$ of $X_{E,P}/X_{E,P}$ such that, for $\mathcal{I}$ the ideal sheaf $\mathcal{I}^n/\mathcal{I}^{n+1}$, is locally free with non-negative slopes. We will denote an object of this category as $(\mathcal{X}/X_{E,P}, P/\mathrm{Spd}\mathbb{F}_q)$, or simply $\mathcal{X}/X_{E,P}$, or even simply $\mathcal{X}$ when it will not cause too much confusion.

Remark 2.1.1. All of our methods and definitions apply also to the larger category $X_{E,\square}^{\mathrm{lf}}$ of arbitrary finite locally free thickenings, and in the body of the text we will often work in this setting. However, at a few key points it is important to impose some type of slope condition on the ideal sheaf in order to ensure the semistable locus is open in a suitable sense in the inscribed version of the moduli stack of G-bundles (see Proposition 6.4.3); for example the comparison between inscribed structures on infinite level and finite level local Shimura varieties is weaker without this constraint. For simplicity we have thus assumed this slope condition through the introduction.

Remark 2.1.2. By definition, a finite locally free thickening $\mathcal{X}/X_{E,P}$ is in particular a thickening of $X_{E,P}$ equipped with a structure morphism $\mathcal{X} \to X_{E,P}$. We will make use of this structure morphism at several points, especially in our study of Hodge period maps. It may be conceptually helpful to note, however, that this structure morphism can be viewed as a property of a thickening rather than as an additional choice of data: if such a structure morphism exists then it is unique because there are no continuous derivations on a basis of affinoid opens of $X_{E,P}$ (which is preperfectoid).

The category $X_{E,\square}^{\mathrm{lf}^+}$ is fibered over Perf, thus inherits a v-topology. An *(*lf$^+$*−)inscribed v-sheaf* is a v-sheaf $\mathcal{S}$ on $X_{E,\square}^{\mathrm{lf}^+}$ that sends certain simple

[2] In the body we will work over *affinoid* perfectoid spaces to facilitate the study of moduli problems defined via algebraic Fargues-Fontaine curves. The theories are equivalent.

coproducts to products (see Definition 4.1.3). We will also use inscribed v-stacks, defined similarly.

For $\mathcal{S}$ an inscribed v-sheaf, we write $\mathcal{S}_0$ for the underlying v-sheaf on Perf, $\mathcal{S}_0(P) = \mathcal{S}(X_{E,P}/X_{E,P})$. An inscription on a v-sheaf S is the choice of an inscribed v-sheaf $\mathcal{S}$ and an identification $\mathcal{S}_0 = S$.

Remark 2.1.3. We can view $\mathcal{S}_0$ also as an inscribed v-sheaf with the trivial inscription by $\mathcal{S}_0(\mathcal{X}/X_{E,P}) = \mathcal{S}(X_{E,P}/X_{E,P})$. In light of Remark 2.1.2, one can view this version of $\mathcal{S}_0$ as the de Rham stack of $\mathcal{S}$. We are thus in a somewhat unusual situation from the classical perspective where the natural morphism $\mathcal{S} \to \mathcal{S}_0$ to the de Rham stack admits a canonical section $\mathcal{S}_0 \to \mathcal{S}$. The existence of this canonical section explains why we will be able to make interesting global statements about Hodge period morphisms.

The second equality in Eq. (2.1.0.1) suggests an inscription $E^{\diamond\mathrm{lf}}$ on $\underline{E}$:

$$E^{\diamond\mathrm{lf}}(\mathcal{X}/X_{E,P}) \coloneqq H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

Remark 2.1.4. In the notation $E^{\diamond\mathrm{lf}}$, we are viewing E as a p-adic manifold in the sense of Serre (see, e.g., [21]), just as in the notation $\underline{E}$ one views E as a topological space.

For $\mathcal{S}$ an inscribed v-sheaf, we define its tangent bundle $T_{\mathcal{S}}$ by

$$T_{\mathcal{S}}(\mathcal{X}/X_P) = \mathcal{S}(\mathcal{X}[\epsilon]/X_P), \text{ for } \mathcal{X}[\epsilon] \coloneqq \mathcal{X} \times_{\mathrm{Spa} E} \mathrm{Spa} E[\epsilon]/\epsilon^2.$$

We show that $T_{\mathcal{S}}$ is again an inscribed v-sheaf, and that the structure map $T_{\mathcal{S}} \to \mathcal{S}$ induced by the closed immersion $\mathcal{X} \hookrightarrow \mathcal{X}[\epsilon]$ admits a natural structure of an $E^{\diamond\mathrm{lf}}$-module over $\mathcal{S}$. Indeed, the action of $E^{\diamond\mathrm{lf}}$ comes from the endomorphisms of $\mathcal{X}[\epsilon]/\mathcal{X}$, while the condition on coproducts gives the abelian group structure just as in the deformation theory of functors.

In particular, the underlying v-sheaf $(T_{\mathcal{S}})_0$ is an E-Vector Bundle over the underlying v-sheaf $\mathcal{S}_0$ that we can think of as a Tangent Bundle to $\mathcal{S}_0$ depending on the choice of inscription (as it should by Example 1.0.2!).

Example 2.1.5. This definition unifies previous constructions of Tangent Bundles for certain v-sheaves (see Section 5.4 for details and generalizations):

(1) Let L/E be a nonarchimedean extension. Recall that $\mathrm{Spd} L$ sends $P \in \mathrm{Perf}$ to the set of untilts $P^\sharp/\mathrm{Spa} L$, and that for any such untilt there is associated a canonical Cartier divisor $\infty : P^\sharp \hookrightarrow X_{E,P}$. We may view $\mathrm{Spd} L$ (or any other v-sheaf) as a *trivially inscribed* v-sheaf sending $\mathcal{X}/X_{E,P}$ to $(\mathrm{Spd} L)(P)$. For any trivially inscribed v-sheaf, its tangent bundle is the trivial bundle.

Now, for Z/L a smooth rigid analytic variety, let $(Z/L)^{\diamond\mathrm{lf}}$ send $\mathcal{X}/X_{E,P}$ to the set of pairs consisting of an untilt $P^\sharp \to \mathrm{Spa} L$ and a map $\mathcal{X} \times_{X_{E,P}} P^\sharp \to Z$ covering $P^\sharp \to \mathrm{Spa} L$. Then $(Z/L)^{\diamond\mathrm{lf}}$ is an inscribed v-sheaf over $\mathrm{Spd} L$ with underlying v-sheaf $(Z^{\diamond\mathrm{lf}})_0 = Z^\diamond$. In this case, $(T_{(Z/L)^{\diamond\mathrm{lf}}})_0 = (T_Z)^\diamond$, where T_Z denotes the usual rigid analytic tangent space of Z. The $\underline{E}$-module structure on the former agrees with the restriction of the $\mathcal{O}_{Z_v}$-module structure on the latter.

(2) For $Z/X_{E,P_0}$ a smooth morphisms of sousperfectoid adic spaces, let $(Z/X_{E,P_0})^{\diamond \mathrm{lf}}$ send $\mathcal{X}/X_{E,P}$ to the set of pairs consisting of a maps $P \to P_0$ and a map $\mathcal{X} \to Z$ covering the induced $X_{E,P} \to X_{E,P_0}$. Then $(Z/X_{E,P_0})^{\diamond \mathrm{lf}}$ lies over the trivially inscribed $P_0^\diamond$. The underlying v-sheaf $((Z/X_{P_0})^{\diamond \mathrm{lf}})_0$ is the moduli of sections $\mathcal{M}_Z$ of the Fargues-Scholze Jacobian criterion for cohomological smoothness [7, IV.4] and $(T_{(Z/X_{E,P_0})^{\diamond \mathrm{lf}}})_0$ is the tangent bundle implicit therein (cf. [14]).

Remark 2.1.6. As a preliminary definition, one could say an inscribed diamond (resp. inscribed locally spatial diamond, etc.) is an inscribed v-sheaf $\mathcal{S}$ such that $\mathcal{S}_0$ is a diamond (resp. locally spatial diamond, etc.). It may be better to enforce a stronger finiteness condition that controls also the values on thickenings; in particular, it could be reasonable to impose a condition that ensures the tangent bundles satisfy the finiteness condition of being (inscribed) relative Banach-Colmez spaces. We leave the question of such a definition to the side for now; for all of our computations below it will suffice to work only with the notion of an inscribed v-sheaf or v-stack, and all of the tangent bundles we compute in specific examples will be so nice that they must furnish examples for any reasonable future definition.

Example 2.1.7 (Period sheaves)**.** Given a pair $(\mathcal{X}/X_P, P/\mathrm{Spd}E)$, where $P/\mathrm{Spd}E$ corresponds to an untilt $P^\sharp/E$, there is a canonical Cartier divisor ∞ on $\mathcal{X}$ associated to $P^\sharp$ (the pullback along $\mathcal{X} \to X_P$ of the usual Cartier divisor $\infty : P^\sharp \hookrightarrow X_P$ as in Example 2.1.5-(1)). We thus obtain natural inscribed period sheaves $\mathcal{O}$ (resp. $\mathbb{B}_e$, resp. $\mathbb{B}_{\mathrm{dR}}^+$, resp. $\mathbb{B}_{\mathrm{dR}}$) over $\mathrm{Spd}E$ (with its trivial inscription) sending $(\mathcal{X}/X_P, P/\mathrm{Spd}E)$ to the functions on ∞ (resp. meromorphic functions on $\mathcal{X} - \infty$, resp. functions on the formal neighborhood of ∞ in $\mathcal{X}$, resp. meromorphic functions on the formal neighborhood of ∞ in $\mathcal{X}$). Both $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathbb{B}_{\mathrm{dR}}$ are filtered by the order of the zero or pole at ∞. These are inscribed $E^{\diamond \mathrm{lf}}$-algebras whose underlying v-sheaves recover the usual period sheaves of the same names.

2.2. **The B_{dR}^+-affine Grassmannian.** The Schubert cells in the B_{dR}^+-affine Grassmannian attached to a connected linear algebraic group G/E are the fundamental period domains in p-adic Hodge theory. In [23] (in the reductive case) and [13] (in general) they are constructed as diamonds, along with Bialynicki-Birula maps to corresponding partial flag varieties. We now explain how to inscribe them and differentiate the Bialynicki-Birula map.

For G/E a connected linear algebraic group, $G(\mathbb{B}_{\mathrm{dR}})$ is again an inscribed v-sheaf, and it admits natural Schubert cells $C_{[\mu]} \subseteq G(\mathbb{B}_{\mathrm{dR}}) \times_{\mathrm{Spd}E} \mathrm{Spd}E([\mu])$ parameterized by the geometric conjugacy classes $[\mu]$ of cocharacters of G (here $E([\mu])$ is the reflex field, i.e. the field of definition of $[\mu]$). When G is reductive, they form a stratification of the underlying v-sheaf $(G(\mathbb{B}_{\mathrm{dR}}))_0$. We can then define inscribed versions of the B_{dR}^+-affine Grassmannian for G and its Schubert cells[3] by $\mathrm{Gr}_G := G(\mathbb{B}_{\mathrm{dR}})/G(\mathbb{B}_{\mathrm{dR}}^+)$ and $\mathrm{Gr}_{[\mu]} := C_{[\mu]}/G(\mathbb{B}_{\mathrm{dR}}^+)$.

[3] In fact in the body it will be slightly more convenient to define $\mathrm{Gr}_{[\mu]}$ before $C_{[\mu]}$.

Remark 2.2.1. These definitions make sense if we replace $X_{E,\Box}^{\mathrm{lf}^+}$ with the category of finite locally free thickenings of the formal neighborhood of the canonical Cartier divisor $\Box^\sharp \hookrightarrow X_{E,\Box}$. Such a framework has the advantage that all tangent bundles are $\mathbb{B}_{\mathrm{dR}}^+$-modules and derivatives are $\mathbb{B}_{\mathrm{dR}}^+$-linear, thus this is how we develop the material in Section 7. However, to compare with other moduli spaces, one should only evaluate on the restrictions of thickenings of the associated Fargues-Fontaine curve, where one obtains the definitions above; since this is our main interest, we have streamlined the statement of our results in Section 2 by using this framework throughout.

There is a natural inscribed Bialynicki-Birula map $\mathrm{BB} : \mathrm{Gr}_{[\mu]} \to \mathrm{Fl}_{[\mu^{-1}]}^{\diamond\mathrm{lf}}$, where the target is the inscribed rigid analytic flag variety as in Example 2.1.5-(1): Associated to any representation V of G there is a natural locally free $\mathbb{B}_{\mathrm{dR}}^+$-module V_{univ}^+ over Gr_G equipped with a trivialization

$$V_{\mathrm{univ}}^+ \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathbb{B}_{\mathrm{dR}} = V \otimes_E \mathbb{B}_{\mathrm{dR}},$$

and the filtration on $V \otimes_E \mathcal{O}$ used to give a Tannakian definition of BB is

$$\begin{aligned}\mathrm{Fil}^i_{V_{\mathrm{univ}}^+}(V \otimes_E \mathcal{O}) &= \frac{(\mathrm{Fil}^i \mathbb{B}_{\mathrm{dR}}^+ \cdot V_{\mathrm{univ}}^+) \cap V \otimes \mathbb{B}_{\mathrm{dR}}^+}{(\mathrm{Fil}^i \mathbb{B}_{\mathrm{dR}}^+ \cdot V_{\mathrm{univ}}^+) \cap V \otimes \mathrm{Fil}^1 \mathbb{B}_{\mathrm{dR}}^+} \\ &\subseteq V \otimes_E \mathcal{O} = (V \otimes_E \mathbb{B}_{\mathrm{dR}}^+)/(V \otimes_E \mathrm{Fil}^1 \mathbb{B}_{\mathrm{dR}}^+).\end{aligned}$$

Theorem A. *Let $\mathfrak{g} = \mathrm{Lie}\, G$ with the adjoint action of G. Differentiating the action of $G(\mathbb{B}_{\mathrm{dR}}^+)$ on $\mathrm{Gr}_{[\mu]}$ at the identity element induces an identification*

$$T_{\mathrm{Gr}_{[\mu]}} = \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^+ / (\mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^+ \cap \mathfrak{g}_{\mathrm{univ}}^+) = (\mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^+ + \mathfrak{g}_{\mathrm{univ}}^+)/\mathfrak{g}_{\mathrm{univ}}^+.$$

It fits in a natural commutative diagram

$$\begin{array}{ccc} T_{\mathrm{Gr}_{[\mu]}} & = & \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^+ / \left(\mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^+ \cap \mathfrak{g}_{\mathrm{univ}}^+\right) \\ & & \downarrow\!\!\downarrow \\ \Big\downarrow dBB & & \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^+ / \left(\mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^+ \cap \mathfrak{g}_{\mathrm{univ}}^+ + \mathfrak{g} \otimes_E \mathrm{Fil}^1 \mathbb{B}_{\mathrm{dR}}^+\right) \\ & & \| \\ \mathrm{BB}^* T_{\mathrm{Fl}_{[\mu^{-1}]}} & = & (\mathfrak{g} \otimes_E \mathcal{O}) / \mathrm{Fil}^0_{\mathfrak{g}_{\mathrm{univ}}^+} (\mathfrak{g} \otimes_E \mathcal{O}) \end{array}$$

Proof. This combines Proposition 7.2.3 and Theorem 7.3.1. $\Box$

Remark 2.2.2. It follows from the definition of the filtrations that the subbundle of $T_{\mathrm{Gr}_{[\mu]}}$ annihilated by $\mathrm{Fil}^1 \mathbb{B}_{\mathrm{dR}}^+$ maps under $d\mathrm{BB}$ to

$$\frac{\mathrm{Fil}^{-1}_{\mathfrak{g}_{\mathrm{univ}}^+}(\mathfrak{g} \otimes_E \mathcal{O})}{\mathrm{Fil}^{0}_{\mathfrak{g}_{\mathrm{univ}}^+}(\mathfrak{g} \otimes_E \mathcal{O})},$$

i.e. the part of $T_{\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu^{-1}]}}$ where the universal filtration satisfies Griffiths transversality for the trivial connection. Since Hodge/Grothendieck-Messing period maps for de Rham local systems on smooth rigid analytic varieties factor through $\mathrm{Gr}_{[\mu]}$, but the tangent bundles of smooth rigid analytic varieties are $\mathcal{O}$-modules and thus annihilated by $\mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}$, this gives a conceptual explanation of the Griffiths transversality on the associated filtered vector bundle with connection (see Corollary 7.3.2 and Remark 7.3.3).

2.3. **Moduli of mixed characteristic local shtukas with one leg.** Let $\overline{E}/E$ be an algebraic closure, let $\mathbb{C}_p$ be the completion of $\overline{E}$ for the unique extension of the p-adic absolute value, and let $\breve{E} \subseteq \mathbb{C}_p$ be the completion of the maximal unramified extension of E. For G/E a connected linear algebraic group, $b \in G(\breve{E})$, and $[\mu]$ a conjugacy class of cocharacters of $G_{\overline{E}}$, we now explain how to define the moduli of mixed characteristic local shtuka with one leg $\mathscr{M}_{b,[\mu]}$ as an inscribed v-sheaf over $\mathrm{Spd}\breve{E}([\mu])$. To make the definition, we first recall that the element b gives rise to a canonical G-torsor $\mathcal{E}_b$ on any $X_{E,P}$, and thus by pullback also on $\mathcal{X}/X_{E,P}$, and that there is a canonical trivialization

$$\mathrm{triv}_b : \mathcal{E}_1|_{\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}} \xrightarrow{\sim} \mathcal{E}_b|_{\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}}.$$

We define $\mathscr{M}_{b,[\mu]}(\mathcal{X}/X_P, P/\mathrm{Spd}\breve{E}([\mu]))$ to be the set of meromorphic isomorphisms (i.e. modifications) $\varphi : \mathcal{E}_1|_{\mathcal{X}\setminus\infty} \to \mathcal{E}_b|_{\mathcal{X}\setminus\infty}$ whose period matrix

$$c_{\mathrm{dR}}(\varphi) \coloneqq \mathrm{triv}_b^{-1} \circ \varphi \in G(\mathbb{B}_{\mathrm{dR}})$$

lies in the Schubert cell $C_{[\mu]} \subseteq G(\mathbb{B}_{\mathrm{dR}})$. Extending the non-inscribed definitions (cf. [13, §8]), there are natural Hodge and Hodge-Tate filtration period maps to the inscribed partial flag varieties $\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu^{\mp 1}]}$ as in Example 2.1.5,

$$\pi_{\mathrm{Hdg}} : \mathscr{M}_{b,[\mu]} \to \mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu^{-1}]} \text{ and } \pi_{\mathrm{HT}} : \mathscr{M}_{b,[\mu]} \to \mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu]}.$$

They are refined along the Bialynicki-Birula maps by lattice period maps[4]

$$\pi_1 : \mathscr{M}_{b,[\mu]} \to \mathrm{Gr}_{[\mu]} \text{ and } \pi_2 : \mathscr{M}_{b,[\mu]} \to \mathrm{Gr}_{[\mu^{-1}]}.$$

We write $\tilde{G}_b$ for the inscribed v-sheaf of automorphisms of $\mathcal{E}_b$ and note that $G(E^{\diamond\mathrm{lf}})$ is the inscribed v-sheaf of automorphisms of $\mathcal{E}_1$. Thus there are action maps

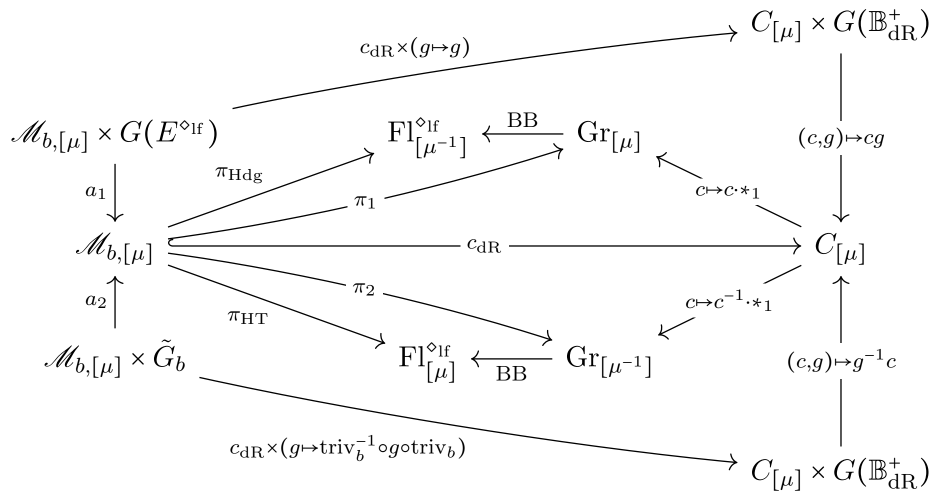
$$a_1 : \mathscr{M}_{b,[\mu]} \times G(E^{\diamond\mathrm{lf}}) \to \mathscr{M}_{b,[\mu]}, (\varphi, g) \mapsto (\varphi \circ g)$$
$$\text{and } a_2 : \mathscr{M}_{b,[\mu]} \times \tilde{G}_b \to \mathscr{M}_{b,[\mu]}, (\varphi, g) \mapsto (g^{-1} \circ \varphi).$$

In the following $*_1 : \mathrm{Spd}\breve{E} \to \mathrm{Gr}_G$ denotes the natural base-point $G(\mathbb{B}^+_{\mathrm{dR}})$.

Theorem B. *Let $E/\mathbb{Q}_p$ be a finite extension, let G/E be a connected linear algebraic group, let $b \in G(\breve{E})$, and let $[\mu]$ be a conjugacy class of cocharacters*

[4]In the notation of [13], π_1 is $\pi_{\mathcal{L}_{\text{ét}}}$ and π_2 is $\pi_{\mathcal{L}_{\mathrm{dR}}}$.

of $G_{\overline{E}}$. Then $\mathscr{M}_{b,[\mu]}$ is an inscribed v-sheaf over $\mathrm{Spd}\breve{E}([\mu])$ and the following diagram commutes:



The group actions realize π_1 as a $\tilde{G}_b$-equivariant $G(E^{\diamond \mathrm{lf}})$-torsor over its image $\mathrm{Gr}^{b\text{-adm}}_{[\mu]}$, the inscribed b-admissible locus, and π_2 as a $G(E^{\diamond \mathrm{lf}})$-equivariant $\tilde{G}_b$-torsor over its image $\mathrm{Gr}^{[b]}_{[\mu^{-1}]}$, the inscribed Newton stratum.

Proof. This is Theorem 9.2.1 for $b_1 = 1$ and $b_2 = b$. □

There is an unbounded version $\mathscr{M}_b$ of the moduli functor such that $\mathscr{M}_{b,[\mu]} = \mathscr{M}_b \times_{G(\mathbb{B}_{\mathrm{dR}})} C_{[\mu]}$. Since inscribed v-sheaves are preserved under fiber products, that $\mathscr{M}_{b,[\mu]}$ is an inscribed v-sheaf follows from the same property for $C_{[\mu]}$, $G(\mathbb{B}_{\mathrm{dR}})$, and $\mathscr{M}_b$. That $C_{[\mu]}$ and $G(\mathbb{B}_{\mathrm{dR}})$ are inscribed v-sheaves is established in our study of the B^+_{dR} affine Grassmannians, and that $\mathscr{M}_b$ is an inscribed v-sheaf is a consequence of the fact that the moduli stack of G-bundles on $\mathcal{X}$ is an inscribed v-stack (established in Section 6). The rest of Theorem B is then immediate by tracing through the definitions; the unbounded analog is given in Theorem 9.1.5.

With Theorem B in place, we compute the tangent bundles of the various spaces appearing in a way that makes the derivatives of these maps explicit. To explain our computation, we introduce more notation! We let $\mathfrak{g} := \mathrm{Lie}\, G(E)$ and write $\mathfrak{g}_b$ for the isocrystal arising from b and the adjoint representation on $\mathfrak{g}$. Associated to $\mathfrak{g}_b$ there is a vector bundle $\mathcal{E}(\mathfrak{g}_b)$ on $\mathcal{X}$ over $\mathrm{Spd}\breve{E}$ (obtained equivalently as the push-out of $\mathcal{E}_b$ along the adjoint representation). We write $\mathrm{BC}(\mathfrak{g}_b)$ for its global sections and $\mathrm{BC}(\mathfrak{g}_b[1])$ for the v-sheafification of its first cohomology. These are inscribed $E^{\diamond \mathrm{lf}}$-modules over $\mathrm{Spd}\breve{E}$; more generally, given a vector bundle $\mathcal{E}$ on $\mathcal{X}$ over an inscribed v-sheaf $\mathcal{S}$, we write $\mathrm{BC}(\mathcal{E})$ for its global sections and $\mathrm{BC}(\mathcal{E}[1])$ for the v-sheafification of its first cohomology. These are inscribed $E^{\diamond \mathrm{lf}}$-modules over $\mathcal{S}$ generalizing the underlying v-sheaf constructions in [7, Chapter II].

For any inscribed group H, we write $\mathrm{Lie}\, H$ for its tangent space at the identity. In particular,

$$\mathrm{Lie}(G(E^{\diamond\mathrm{lf}})) = \mathfrak{g} \otimes_E E^{\diamond\mathrm{lf}} = \mathrm{BC}(\mathfrak{g}_1) \text{ and } \mathrm{Lie}\, \tilde{G}_b = \mathrm{BC}(\mathfrak{g}_b).$$

Theorem C. *The* $\mathbb{B}_{\mathrm{dR}}^+$*-module on* $\mathscr{M}_{b,[\mu]}$

$$\mathfrak{g}_{\max}^+ \coloneqq \mathfrak{g}_b \otimes_{\breve{E}} \mathbb{B}_{\mathrm{dR}}^+ + \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^+ \subseteq \mathfrak{g}_b \otimes_{\breve{E}} \mathbb{B}_{\mathrm{dR}} =^{\varphi} \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}$$

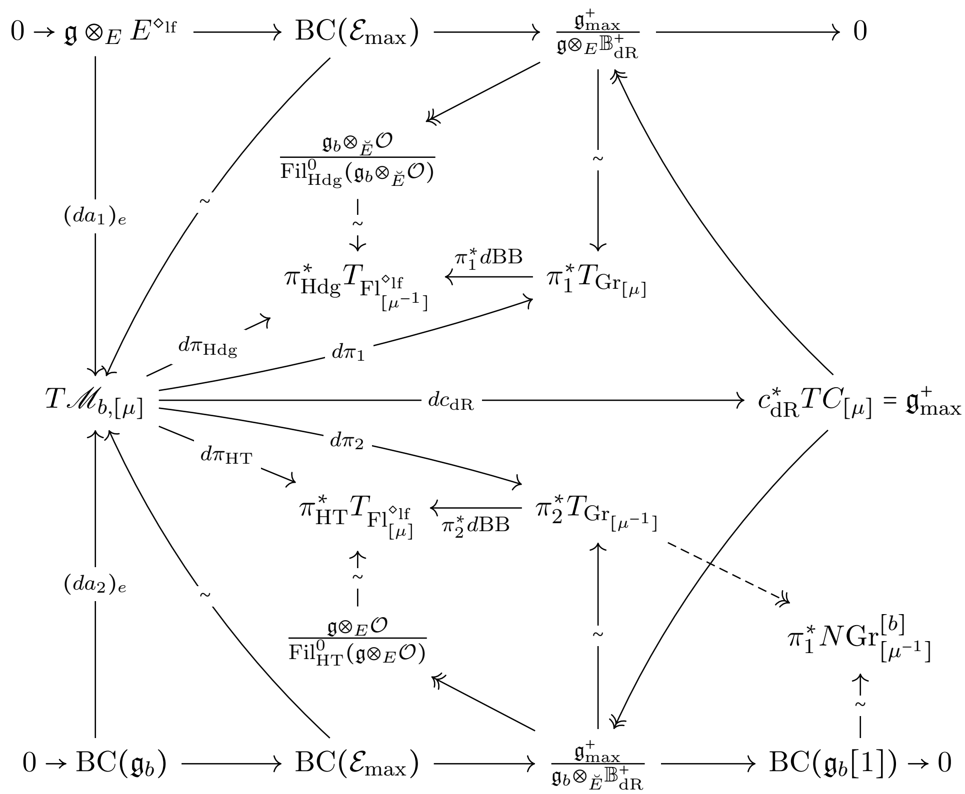
is locally free. In particular, there is a vector bundle $\mathcal{E}_{\max}$ *on* $\mathcal{X}$ *over* $\mathscr{M}_{b,[\mu]}$ *fitting into two canonical modification exact sequences of sheaves on* $\mathcal{X}$

$$0 \to \mathfrak{g} \otimes_E \mathcal{O}_{\mathcal{X}} \to \mathcal{E}_{\max} \to \infty_* \left(\mathfrak{g}_{\max}^+ / \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^+\right) \to 0 \tag{2.3.0.1}$$

$$\text{and } 0 \to \mathcal{E}(\mathfrak{g}_b) \to \mathcal{E}_{\max} \to \infty_* \left(\mathfrak{g}_{\max}^+ / \mathfrak{g}_b \otimes_{\breve{E}} \mathbb{B}_{\mathrm{dR}}^+\right) \to 0. \tag{2.3.0.2}$$

There is a natural isomorphism $c_{\mathrm{dR}}^* T_{C_{[\mu]}} = \mathfrak{g}_{\max}^+$ *induced by the action of* $G(\mathbb{B}_{\mathrm{dR}}^+) \times G(\mathbb{B}_{\mathrm{dR}}^+)$ *on* $C_{[\mu]}$ *and a unique isomorphism* $\mathrm{BC}(\mathcal{E}_{\max}) \xrightarrow{\sim} T_{\mathscr{M}_{b,[\mu]}}$ *whose composition with* dc_{dR} *is the natural map* $\mathrm{BC}(\mathcal{E}_{\max}) \to \mathfrak{g}_{\max}^+$ *given by restricting global sections to* $\mathrm{Spec}\, \mathbb{B}_{\mathrm{dR}}^+$*. These isomorphisms fit into the following commutative diagram of inscribed* $E^{\diamond\mathrm{lf}}$*-modules on* $\mathscr{M}_{b,[\mu]}$*, where:*

- *the top row is the exact sequence induced by* v*-sheafifying the long exact sequence of cohomology of Eq. (2.3.0.1),*
- *the bottom row is the exact sequence induced by* v*-sheafifying the long exact sequence of cohomology of Eq. (2.3.0.2),*
- *isomorphisms involving* $T_{\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu^{\pm 1}]}}$ *and* $T_{\mathrm{Gr}_{[\mu^{\mp 1}]}}$ *are from Theorem A,*
- *and* $N_{\mathrm{Gr}^{[b]}_{[\mu^{-1}]}}$ *denotes the normal bundle of* $\mathrm{Gr}^{[b]}_{[\mu^{-1}]} \subseteq \mathrm{Gr}_{[\mu^{-1}]}$*.*



Proof. This combines Corollary 9.2.3 and Corollary 9.2.4 in the case $b_1 = 1$ and $b_2 = b$, and adds in the computation of $d\mathrm{BB}$ in Theorem A. The 0 in the top right is obtained from Proposition 6.3.2. □

Remark 2.3.1. Morally, no normal bundle appears in the top part of the diagram of Theorem C because the underlying v-sheaf of the admissible locus is open. Practically, no normal bundle appears because $\mathrm{BC}(\mathfrak{g}_1[1]) = 0$. When b is basic, the underlying v-sheaf of the associated Newton stratum is also open. In that case, we also have $\mathrm{BC}(\mathfrak{g}_b[1]) = 0$, so that the normal bundle also vanishes in the bottom part (morally and practically).

To prove Theorem C, we first establish in Corollary 9.1.6 an analogous result for the unbounded moduli space $\mathscr{M}_b$, where it is the fundamental exact sequences of p-adic Hodge theory associated to the vector bundles $\mathfrak{g} \otimes_E \mathcal{O}_{\mathcal{X}}$ and $\mathcal{E}(\mathfrak{g}_b)$ that appear in the top and bottom rows. This is possible because the unbounded moduli space admits a simple structure as a bitorsor over $\mathrm{Spd}\breve{E}$ (see Theorem 9.2.1) from which we can extract this description of its tangent bundle and the derivatives. We then use the period maps to cut-out the diagram of Theorem C; the inscribed formalism allows us to make this computation as one would expect, e.g., in classical differential topology.

Implicit in the statement of Theorem C is also a simple description of the tangent and normal bundles of the inscribed Newton strata. One might also ask for simple descriptions of the generalized Newton strata which form the boundary of the admissible locus. When b is basic, these can be reinterpreted as Newton strata for different groups by the two towers isomorphism; in fact, even if b is not basic this applies if one allows more general group schemes over the Fargues-Fontaine curve in the definitions (see Remark 2.3.3). However, a simpler way this can be addressed is by allowing more general moduli of modifications from $\mathcal{E}_{b_1}$ to $\mathcal{E}_{b_2}$ for any b_1, b_2 in $G(\breve{E})$, and then examining the resulting structure of the images of their period maps. Our methods apply equally well in this more general context, and in the body of the text we give all of our results in this setting. In particular, the description of tangent and normal bundles of generalized Newton strata is Corollary 9.2.4.

Remark 2.3.2. One can hope that these descriptions of the tangent and normal bundles of the generalized Newton stratifications will have applications, e.g., to a Banach-Colmez theory of holonomic D-modules.

Remark 2.3.3. In Section 9.3 we explain a two towers isomorphism in the inscribed setting that is valid even without the basic hypothesis: at the level of c_{dR} it is given simply by $c \mapsto c^{-1}$, so that it extends to an automorphism of the diagrams in Theorem B and Theorem C that reflects the objects across the horizontal axis traced by c_{dR}. The description in this level of generality appears to be new even on the underlying v-sheaf. Similarly, the description of $\mathscr{M}_b$ as a bitorsor over $\mathrm{Spd}\breve{E}$, though immediate from the definition, does not seem to have been utilized previously even for the underlying v-sheaf.

Remark 2.3.4 (Finite level local Shimura varieties)**.** For $[\mu]$ minuscule, i.e. the local Shimura case, and $K_p \leq G(\mathbb{Q}_p)$ compact open, it is shown in [25] that there exist rigid analytic local Shimura varieties $\mathscr{M}^{\mathrm{rig}}_{b,[\mu],K_p}$ representing the diamonds $(\mathscr{M}_{b,[\mu]})_0/K_p$. Indeed, in the minuscule case the Bialynicki-Birula map is an isomorphism $(\mathrm{Gr}_{[\mu]})_0 = \mathrm{Fl}^{\diamond}_{[\mu^{-1}]}$, so that one can construct $\mathscr{M}^{\mathrm{rig}}_{b,[\mu],K_p}$ by invoking the equivalence of étale sites $(\mathrm{Fl}_{[\mu^{-1}]})_{\text{ét}} = (\mathrm{Fl}^{\diamond}_{[\mu^{-1}]})_{\text{ét}}$.

We claim that, in this minuscule case, there is a canonical identification of $\mathscr{M}_{b,[\mu],K_p} := \mathscr{M}_{b,[\mu]}/K_p^{\diamond\mathrm{lf}}$ with $(\mathscr{M}^{\mathrm{rig}}_{b,[\mu],K_p})^{\diamond\mathrm{lf}}$ where the latter is defined via Example 2.1.5. To see this, the key point is to construct the map from $\mathscr{M}_{b,[\mu]}$ to $(\mathscr{M}^{\mathrm{rig}}_{b,[\mu],K_p})^{\diamond\mathrm{lf}}$. This is straightforward: to obtain it, we first note that we are given the inscribed period map π_{Hdg} to $\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu^{-1}]}$ and a lift of $(\pi_{\mathrm{Hdg}})_0$ to the underlying v-sheaf $((\mathscr{M}^{\mathrm{rig}}_{b,[\mu],K_p})^{\diamond\mathrm{lf}})_0$. Then, since the maps $\mathcal{M}^{\mathrm{rig}}_{b,[\mu],K_p} \to \mathrm{Fl}_{[\mu^{-1}]}$ are étale, there is a unique deformation of the map on the underlying v-sheaves that is compatible with π_{Hdg}. That it is an isomorphism follows from Proposition 6.4.3, which uses crucially the slope condition in the definition of $X^{\mathrm{lf}^+}_{E,\square}$ (see Remark 2.1.1).

The comparison between the infinite level inscription and finite level inscriptions in the case of *global* Shimura varieties is based on the same idea, but, as we will explain in Section 2.4, the analysis of the Hodge period map is more delicate.

2.4. **Inscribed global Shimura varieties.** Let (G, X) be a Shimura datum, and assume the maximal $\mathbb{R}$-split $\mathbb{Q}$-anisotropic central torus is equal to G (this occurs, e.g., if (G, X) is of Hodge type; since the cases where Conjecture 2.4.1 below is currently known all fall under this umbrella, it is a reasonable simplifying assumption to impose).

We fix a p-adic field L containing the reflex field, i.e. the field of definition of the conjugacy class of Hodge cocharacters $[\mu]$. For $K \leq G(\mathbb{A}^\infty)$ a sufficiently small compact open, we write Sh_K for the associated Shimura variety of level K as a (smooth) rigid analytic variety over L. For $K^p \leq G(\mathbb{A}^{\infty p})$ compact open, we write $\mathrm{Sh}^\diamond_{K^p} := \varprojlim_{K_p} \mathrm{Sh}^\diamond_{K_pK^p}$, where the limit is over compact open subgroups $K_p \leq G(\mathbb{Q}_p)$ such that K_pK^p is sufficiently small. There is an action of $G(\mathbb{Q}_p)$ on $\mathrm{Sh}^\diamond_{K^p}$ and a $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period map $\pi^\diamond_{\mathrm{HT}} : \mathrm{Sh}^\diamond_{K^p} \to \mathrm{Fl}^\diamond_{[\mu]}$. We write $\pi^\diamond_{K_p} : \mathrm{Sh}^\diamond_{K^p} \to \mathrm{Sh}^\diamond_{K_pK^p}$ for the natural projection map — it is a torsor for $\underline{K_p}$.

We want to define an inscription on $\overline{\mathrm{Sh}^\diamond_{K^p}}$. To do so, we will need to assume a version of the Igusa stacks fiber product conjecture of Scholze for the minimal compactification (see [29, 5]). To state this conjecture, we recall that, for any sufficiently small K as above, there is a minimal compactification $\mathrm{Sh}_K \subseteq \mathrm{Sh}^*_K$. We write $(\mathrm{Sh}^*_{K^p})^\diamond = \varprojlim_{K_p} (\mathrm{Sh}^*_{K_pK^p})^\diamond$. We write $\mathrm{Bun}G$ for the v-stack of G-bundles on the Fargues-Fontaine curve and $\mathrm{BL} : \mathrm{Fl}^\diamond_{[\mu]} \to \mathrm{Bun}G$ for the map obtained by making the Beauville-Laszlo modification of the trivial bundle along the $\mathbb{B}^+_{\mathrm{dR}}$-lattice associated to the minuscule filtration parameterized by $\mathrm{Fl}^\diamond_{[\mu]}$.

Conjecture 2.4.1 (see Conjecture 1.1-(3) of [29])**.**

(1) *For any K^p, the Hodge-Tate period map extends to $\pi^\diamond_{\mathrm{HT}} : (\mathrm{Sh}^*_{K^p})^\diamond \to \mathrm{Fl}^\diamond_{[\mu]}$.*

(2) *As K^p varies, there is a compatible family of small v-stacks $\mathrm{Igs}^*_{K^p}$ with maps*

$$\overline{\pi}_{\mathrm{HT}} : \mathrm{Igs}^*_{K^p} \to \mathrm{Bun}_G \text{ and } q_{K^p} : (\mathrm{Sh}^*_{K^p})^\diamond \to \mathrm{Igs}^*_{K^p}$$

such that $\overline{\pi}_{\mathrm{HT}}$ is 0-truncated and, for each K^p, the following diagram is Cartesian:

(2.4.1.1)
$$\begin{array}{ccc} (\mathrm{Sh}^*_{K^p})^\diamond & \xrightarrow{\pi^\diamond_{\mathrm{HT}}} & \mathrm{Fl}^\diamond_{[\mu]} \\ \downarrow{\scriptstyle q_{K^p}} & \lrcorner & \downarrow{\scriptstyle \mathrm{BL}} \\ \mathrm{Igs}^*_{K^p} & \xrightarrow{\overline{\pi}_{\mathrm{HT}}} & \mathrm{Bun}G \end{array}$$

Conjecture 2.4.1 is known in most PEL cases by [29, Theorem 1.3]. It is known in the compact Hodge type case by [5, Theorem I].

Remark 2.4.2. The results of [5] prove the existence in the general Hodge type case of a cartesian diagram as in Eq. (2.4.1.1) over the good reduction locus $(\mathrm{Sh}^{\circ}_{K^p})^{\diamond} \subseteq \mathrm{Sh}^{\diamond}_{K^p} \subseteq (\mathrm{Sh}^{*}_{K^p})^{\diamond}$. When the Shimura variety is compact, these inclusions are equalities. In the general Hodge type case, one can still use this result to carry out our constructions over the good reduction locus of the infinite level Shimura variety; we do not state this here, but the interested reader can adapt the statements below.

Assumption 2.4.3. For the remainder of this section, we assume Conjecture 2.4.1 holds for any Shimura variety under discussion.

We will now define $\mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p}$. It will be open in $(\mathrm{Sh}^{*}_{K^p})^{\diamond\mathrm{lf}}$, which we will define by inscribing the two maps $\overline{\pi}_{\mathrm{HT}}$ and BL and then taking their fiber product. A natural inscription on Bun$G$ is $\mathcal{X}^*\mathrm{B}G$, which sends $\mathcal{X}/X_P$ to the groupoid of G-torsors on $\mathcal{X}$, and the same formalism of Beauville-Laszlo glueing extends to a map $\mathrm{BL}^{\mathrm{lf}} : \mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu]} \to \mathcal{X}^*BG$ (where $\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu]}$ is defined as in Example 2.1.5). We equip Igs_{K^p} with the trivial inscription, and we write $\overline{\pi}^{\diamond\mathrm{lf}}_{\mathrm{HT}}$ for the composition of $\overline{\pi}_{\mathrm{HT}}$ with the natural map $\mathrm{Bun}G \to \mathcal{X}^*\mathrm{B}G$ (by pullback of bundles from X_P to $\mathcal{X}$). Then, we define

$$(\mathrm{Sh}^{*}_{K_p})^{\diamond\mathrm{lf}} := (\overline{\pi}_{\mathrm{HT}})^{\diamond\mathrm{lf}} \times_{\mathcal{X}^*\mathrm{B}G} \mathrm{BL}^{\diamond\mathrm{lf}}.$$

Finally, we restrict to the open infinite level Shimura variety:

$$\mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p} := (\mathrm{Sh}^{*}_{K_p})^{\diamond\mathrm{lf}} \times_{(\mathrm{Sh}^{*}_{K^p})^{\diamond}} \mathrm{Sh}^{\diamond}_{K^p}.$$

It follows that $(\mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p})_0 = \mathrm{Sh}^{\diamond}_{K^p}$, i.e. that this is an inscription on $\mathrm{Sh}^{\diamond}_{K^p}$.

Remark 2.4.4. We focus on the infinite level Shimura variety rather than its minimal compactification because at finite level the minimal compactifications are not typically smooth.

By construction, there is an action $a_e : \mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p} \times G(\mathbb{Q}_p^{\diamond\mathrm{lf}}) \to \mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p}$ and a $G(\mathbb{Q}_p^{\diamond\mathrm{lf}})$-equivariant map $\pi^{\diamond\mathrm{lf}}_{\mathrm{HT}} : \mathrm{Sh}^{\diamond\mathrm{lf}}_{K_p} \to \mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu]}$. We now describe the computation of $T_{\mathrm{Sh}^{\diamond\mathrm{lf}}_{K_p}}$ and $d\pi^{\diamond\mathrm{lf}}_{\mathrm{HT}}$, and the relation to finite level Shimura varieties. This will mirror the minuscule case of Theorem C, with maps to finite level inscribed Shimura varieties replacing the map π_{Hdg}; however, it will require a non-trivial argument to show such maps exist (see the discussion after the statement of Theorem D). Before making the statement, we note that on $\mathrm{Sh}^{\diamond\mathrm{lf}}_{K_p}$ we have the trivial bundle $\mathfrak{g}\otimes\mathcal{O}_{\mathcal{X}}$ over $\mathcal{X}$ and the pullback of the universal bundle associated to the adjoint representation over $\mathcal{X}^*\mathrm{B}G$, $\mathfrak{g}_{\mathrm{univ}}$. By the construction of the map $\mathrm{BL}^{\diamond\mathrm{lf}}$, we have fixed an isomorphism between these two bundles after restriction to $\mathcal{X}\backslash\infty$.

Theorem D. *With notation above, suppose as in Assumption 2.4.3 that Conjecture 2.4.1 holds for the Shimura varieties under consideration. Then,*

(1) *For sufficently small compact opens $K_p \leq G(\mathbb{Q}_p)$, there are canonical maps $\pi^{\diamond\mathrm{lf}}_{K_p} : \mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p} \to \mathrm{Sh}^{\diamond\mathrm{lf}}_{K_pK^p}$ (where the codomain is defined as in*

Example 2.1.5) extending the projection maps on the underlying v-sheaves and realizing $\mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p}$ *as a* $K_p^{\diamond\mathrm{lf}}$ *torsor over* $\mathrm{Sh}^{\diamond\mathrm{lf}}_{K_pK^p}$. *These maps are compatible as* K_p *and* K^p *vary.*

(2) *The* $\mathbb{B}^+_{\mathrm{dR}}$*-module on* $\mathrm{Sh}^{\diamond\mathrm{lf}}_{K_p}$

$$\mathfrak{g}^+_{\max} \coloneqq \mathfrak{g}_{\mathrm{univ}} \boxtimes \mathbb{B}^+_{\mathrm{dR}} + \mathfrak{g} \otimes_E \mathbb{B}^+_{\mathrm{dR}} \subseteq \mathfrak{g}_{\mathrm{univ}} \boxtimes \mathbb{B}_{\mathrm{dR}} = \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}$$

is locally free. In particular, there is a vector bundle $\mathcal{E}_{\max}$ *on* $\mathcal{X}$ *over* $\mathrm{Sh}^{\diamond\mathrm{lf}}_{K_p}$ *fitting into two canonical modification exact sequences of sheaves on* $\mathcal{X}$

$$0 \to \mathfrak{g} \otimes_E \mathcal{O}_{\mathcal{X}} \to \mathcal{E}_{\max} \to \infty_* (\mathfrak{g}^+_{\max}/\mathfrak{g} \otimes_E \mathbb{B}^+_{\mathrm{dR}}) \to 0 \tag{2.4.4.1}$$

$$\text{and } 0 \to \mathfrak{g}_{\mathrm{univ}} \to \mathcal{E}_{\max} \to \infty_* (\mathfrak{g}^+_{\max}/\mathfrak{g}_{\mathrm{univ}} \boxtimes \mathbb{B}^+_{\mathrm{dR}}) \to 0. \tag{2.4.4.2}$$

There is a canonical identification $\mathrm{BC}(\mathcal{E}_{\max}) = T_{\mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p}}$ *such that the v-sheafification of the associated long exact sequences of cohomology for Eq. (2.4.4.1) and Eq. (2.4.4.2) are identified with*

$$0 \to \mathfrak{g} \otimes \mathbb{Q}_p^{\diamond\mathrm{lf}} \xrightarrow{da_e} \mathrm{BC}(\mathcal{E}_{\max}) = T_{\mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p}} \xrightarrow{d\pi^{\diamond\mathrm{lf}}_{K_p}} (\pi^{\diamond\mathrm{lf}}_{K_p})^* T_{\mathrm{Sh}^{\diamond\mathrm{lf}}_{K_pK^p}} \to 0 \tag{2.4.4.3}$$

and

(2.4.4.4)

$$0 \to \mathrm{BC}(\mathfrak{g}_{\mathrm{univ}}) \to \mathrm{BC}(\mathfrak{g}_{\max}) = T_{(\mathrm{Sh}^{\circ}_{K^p})^{\diamond\mathrm{lf}}} \xrightarrow{d\pi^{\diamond\mathrm{lf}}_{\mathrm{HT}}} (\pi^{\diamond\mathrm{lf}}_{\mathrm{HT}})^* T_{\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu]}} \to \mathrm{BC}(\mathfrak{g}_{\mathrm{univ}}[1]) \to 0.$$

Proof. This combines Lemma 11.2.2 and Lemma 11.4.1. □

Remark 2.4.5. The fibers of $\pi^{\diamond\mathrm{lf}}_{\mathrm{HT}}$ are identified, by the definition using a product, with inscribed variants of the Caraiani-Scholze type Igusa varieties $\mathrm{Igs}_{K^p} \times_{\mathrm{Bun}G} (\mathrm{Bun}G \xleftarrow{b} \mathrm{Spd}\breve{\mathbb{Q}}_p)$. In particular, if we look above a point of the flag variety lying in the Newton stratum for $b \in G(\breve{\mathbb{Q}}_p)$, then the restriction of $\mathfrak{g}_{\mathrm{univ}}$ is the bundle associated to the isocrystal $\mathfrak{g}_b$. The kernel $\mathrm{BC}(\mathfrak{g}_b)$ of $d\pi^{\diamond\mathrm{lf}}_{\mathrm{HT}}$ at such a point, which is the tangent bundle of the fiber, can also be viewed as coming from differentiating the action on this Igusa variety of the group $\tilde{G}_b$ of automorphisms of the associated G-bundle on the relative thickened Fargues-Fontaine curve. Note that the term $\mathrm{BC}(\mathfrak{g}_b[1])$ appearing in the restriction of Eq. (11.2.2.3) is zero if and only if b is basic. Its role here is as the normal bundle of the associated Newton stratum on the flag variety (cf. Remark 2.3.1 and the surrounding discussion). In particular, $\pi^{\diamond\mathrm{lf}}_{\mathrm{HT}}$ is a submersion over the open basic locus but nowhere else; in general, it is only a submersion after pullback to a Newton stratum.

As in the proof of Theorem C, for the computation of the tangent bundle in Theorem D the key step is to define an unbounded variant by replacing $\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu]}$ with the inscribed B^+_{dR}-affine Grassmannian Gr_G in the defining product. The extension of the Beauville-Laszlo map to Gr_G has a simple torsor structure, and since Igs_{K^p} is trivially inscribed, we obtain a computation of the tangent bundle of the unbounded variant of $\mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}}$. The computation of $T_{\mathrm{Sh}_{K_p^{\diamond\mathrm{lf}}}}$ is then deduced by pulling back along the inclusion $T_{\mathrm{Fl}_{[\mu]}^{\diamond\mathrm{lf}}} \to T_{\mathrm{Gr}_G}$.

The most delicate part of Theorem C is actually the construction of the maps $\pi_{K_p}^{\diamond\mathrm{lf}}$. As in Remark 2.3.4, since we know the map $\pi_{K_p}^{\diamond}$ on the underlying v-sheaf, we only need to extend it over formal neighborhoods. We can understand the formal neighborhood at finite level using the Hodge period map for the de Rham filtered G-torsor with connection. Indeed, this can be realized in our theory as a map $\pi_{\mathrm{Hdg},K_p}^{\diamond\mathrm{lf}} : \mathrm{Sh}_{K_pK^p}^{\diamond\mathrm{lf}} \to \mathrm{Fl}_{[\mu^{-1}]}/G^{\diamond}$. It is crucial here that the quotient is by $G^{\diamond}$ and not $G^{\diamond\mathrm{lf}}$; this corresponds to the use of the connection to define the Hodge period map with respect to a flat basis. The classical statement that the Hodge period map is an isomorphism on formal neighborhoods becomes the statement that the following diagram is Cartesian:

$$\begin{array}{ccc} \mathrm{Sh}_{K_pK^p}^{\diamond\mathrm{lf}} & \xrightarrow{\pi_{\mathrm{Hdg},K_p}^{\diamond\mathrm{lf}}} & \mathrm{Fl}_{[\mu^{-1}]}^{\diamond\mathrm{lf}}/G^{\diamond} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{K_pK^p}^{\diamond} & \xrightarrow[\pi_{\mathrm{Hdg},K_p}^{\diamond}]{} & \mathrm{Fl}_{[\mu^{-1}]}^{\diamond}/G^{\diamond} \end{array}$$

Thus, to construct $\pi_{K_p}^{\diamond\mathrm{lf}}$, it suffices to construct an infinite level Hodge period map $\pi_{\mathrm{Hdg}}^{\diamond\mathrm{lf}} \to \mathrm{Fl}_{[\mu^{-1}]}^{\diamond\mathrm{lf}}/G^{\diamond}$ that agrees on the underlying v-sheaf with the composition $\pi_{\mathrm{Hdg},K_p}^{\diamond} \circ \pi_{K_p}^{\diamond}$. This is possible because the Hodge filtration arises naturally from the modification construction; the use of the connection at finite level is replaced by the factorization of $\overline{\pi}_{\mathrm{HT}}^{\diamond\mathrm{lf}}$ through Bun_G.

2.5. Differential topology for diamonds I: Perfectoidness.

It is natural to guess that, for C a perfectoid field and $\mathcal{S}$ a sufficiently nice inscribed v-sheaf over $\mathrm{Spd}C$, the underlying v-sheaf $\mathcal{S}_0$ is represented by a perfectoid space over C if, at every geometric point $s : \mathrm{Spa}(C', C'^+) \to \mathcal{S}_0$, its Tangent Space $(T_{\mathcal{S}})_{0,s}$ is represented by a perfectoid space over C'. In particular, applied to quotients of local and global infinite level Shimura varieties, our computations of tangent bundles described above give a simple prediction that unifies several previous results and observations in the literature.

Let Y be either

(1) an infinite level inscribed local Shimura variety $\mathscr{M}_{b,[\mu]}$ (so μ is minuscule and $E = \mathbb{Q}_p$) as in Section 2.3, or
(2) an infinite level inscribed global Shimura variety $\mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}}$ as in Section 2.4.

For the associated group G, we consider any closed subgroup $H \leq G(\mathbb{Q}_p)$ (which is automatically a p-adic Lie group) and let $H^{\diamond\mathrm{lf}} := \underline{H} \times_{\underline{G(\mathbb{Q}_p)}} G(\mathbb{Q}_p^{\diamond\mathrm{lf}})$. We then consider the quotient $Y_H := Y/H^{\diamond\mathrm{lf}}$.

Remark 2.5.1. When H is a compact open subgroup, Remark 2.3.4 and Theorem D-(1) show that the other natural definition of the finite level inscribed space Y_H in these settings agrees with this quotient.

Writing $\mathrm{Lie}\, H =: \mathfrak{h} \leq \mathfrak{g}$, it follows from our general formalism that the tangent bundle of Y_H is $T_Y/\mathfrak{h}\otimes\mathbb{Q}_p^{\diamond\mathrm{lf}}$ (or rather, its natural descent from Y to Y_H). In particular, using the computation of the tangent bundle of Y coming from Theorem C or Theorem D, it is relatively straightforward to identify the points of $(Y_H)_0$ where the associated Tangent Space is perfectoid:

Proposition 2.5.2. *The locus in $|(Y_H)_0|$ whose geometric points have perfectoid Tangent Space is the open pre-image under π_{HT} of the open locus in $\mathrm{Fl}^{\diamond}_{[\mu]}/H$ whose preimage in $\mathrm{Fl}_{[\mu]}$ consists of those points $x : \mathrm{Spa}(C', C'^+) \to \mathrm{Fl}^{\diamond}_{[\mu]}$ such that, for $\mathfrak{u}_x$ the Lie algebra of the unipotent radical of the associated parabolic of G_C (i.e. the stabilizer of x), $\mathfrak{u}_x \cap \mathfrak{h}_C = \{0\}$.*

Remark 2.5.3. In the non-minuscule case of Section 2.3 these quotients should never be perfectoid, which is why we do not consider it here.

The condition of Proposition 2.5.2 gives an after-the-fact conjectural conceptual explanation of several past results, including those of [19, 15] on perfectoidness of quotients of the Lubin Tate tower and the cohomological vanishing results for global Shimura varieties of [4, 3]. A more natural generality for these perfectoidness considerations is the study of quotients of a p-adic manifold fibrations over arbitrary smooth rigid analytic varieties; we thus defer further discussion to [10], where it will be treated in this context.

2.6. **Differential topology for diamonds II: Cohomological smoothness.** Building on the Fargues-Scholze Jacobian criterion [7, §IV.4], it is natural to conjecture that, for a sufficiently natural morphism of inscribed v-sheaves $f : \mathcal{S} \to \mathcal{S}'$, the morphism of underlying v-sheaves $\mathcal{S}_0 \to \mathcal{S}'_0$ is cohomologically smooth if the relative Tangent Bundle (the v-sheaf underlying the kernel of df) at each geometric point is cohomologically smooth. We will now make this expectation explicit for a variant of $\mathscr{M}_{b,[\mu]} \to \mathrm{Spd}\breve{E}([\mu])$, building on work of Ivanov and Weinstein [14].

We suppose G is reductive. There is a natural determinant morphism from $\mathscr{M}_{b,[\mu]}$ to the associated moduli space for the abelianization G^{ab} of G, and we write $\mathscr{M}^{\tau}_{b,[\mu]}/\mathrm{Spd}\mathbb{C}_p$ for a non-empty fiber over a $\mathbb{C}_p$-point τ under this determinant morphism. The reason to work with $\mathscr{M}^{\tau}_{b,[\mu]}/\mathrm{Spd}\mathbb{C}_p$ is simply that a non-empty open in $\mathscr{M}_{b,[\mu]}$ will never be cohomologically smooth if $G^{\mathrm{ab}} \neq 0$ due to a locally profinite set of connected components (on tangent bundles, this is reflected by a summand isomorphic to $E^{\diamond\mathrm{lf}}$).

The description $T_{\mathscr{M}_{b,[\mu]}} = \mathrm{BC}(\mathcal{E}_{\max})$ of Theorem C implies that $T_{\mathscr{M}^{\tau}_{b,[\mu]}} = \mathrm{BC}(\mathcal{E}^{\circ}_{\max})$, where $\mathcal{E}^{\circ}_{\max}$ is constructed by replacing $\mathfrak{g}$ everywhere with $\mathfrak{g}^{\circ}$, the Lie algebra of the derived subgroup G^{der}. We show:

Theorem E. *Let G/E be a connected reductive group. Then, for $x : \mathrm{Spa}(C, \mathcal{O}_C) \to (\mathscr{M}^{\circ}_{b,[\mu]})_0$ a rank one geometric point, $\mathcal{E}^{\circ}_{\max,x}$ is a vector bundle on $X_{E,C}$ with non-negative Harder-Narasimhan slopes. It has strictly positive Harder-Narasimhan slopes if and only if x is isolated in its fiber for $\pi_1 \times \pi_2$, or equivalently*

$$\pi_2^{-1}(\pi_2(x)) \cap \pi_1^{-1}(\pi_1(x)) \subseteq (\mathscr{M}^{\tau}_{b,[\mu]})_0(C) \textit{ is a discrete subspace.}$$

Proof. This is a part of Proposition 10.3.5. □

The appearance of zero as a Harder-Narasimhan slope is equivalent to the existence of a summand of the Tangent Space at x isomorphic to $\underline{E}$; such a summand breaks the cohomological smoothness of the Tangent Space since it introduces a locally profinite set of connected components. If no such summand exists, the Tangent Space is cohomologically smooth in this case. Thus, in light of Theorem E, our heuristic predicts that $(\mathscr{M}^{\circ}_{b,[\mu]})_0/\mathrm{Spd}\mathbb{C}_p$ is cohomologically smooth on the partially proper open locus whose rank one points are those that are discrete in their fibers for $\pi_1 \times \pi_2$.

It follows from Scholze and Weinstein's classification of p-divisible groups [24] that, in the minuscule EL case, this condition holds at a point x exactly when the p-divisible group with EL structure parameterized by x admits no additional endomorphisms. In this case, following Ivanov and Weinstein [14], we are also able to give an alternate construction of $\mathscr{M}^{\tau}_{b,[\mu]}$ as a moduli of sections for a smooth quasi-projective adic space over $X_{E,\mathbb{C}_p^\flat}$ (as in Example 2.1.5). Comparing with this construction, we show the Fargues-Schoze Jacobian criterion of [6] applies in this case. Thus we obtain

Corollary F. *(See Corollary 10.4.4 for the precise statement). For $(G/\mathbb{Q}_p, b, [\mu])$ arising from a minuscule EL datum, $(\mathscr{M}^{\tau}_{b,[\mu]})_0/\mathrm{Spd}\mathbb{C}_p$ is cohomologically smooth on the partially proper open non-special*[5] *locus parameterizing p-divisible groups with EL structure admitting no extra endomorphisms.*

For b basic, Corollary F is essentially the main result of [14]. We can extend beyond the basic case because our description of $\mathcal{E}^{\circ}_{\max}$ is more robust.

Remark 2.6.1. Knowing that the two different constructions of $\mathscr{M}^{\tau}_{b,[\mu]}$ used in the minuscule EL case produce the same underlying v-sheaf does not automatically imply that the two inscriptions agree — as in Example 1.0.2, asking whether two inscriptions agree is akin to asking whether two differentiable structures on the same topological manifold agree. Thus establishing this agreement is a non-trivial step in the proof of Corollary F, but fortunately it is more or less immediate from the construction of [14].

Remark 2.6.2. As in the study of perfectoidness in Section 2.5, one could also consider the question of cohomological smoothness for quotients by

[5] We caution the reader that the notion of non-special appearing here and in [14] is different than the natural Tannakian definition based on [12, 13]; see Remark 10.4.5.

closed subgroups. Similarly, one could also treat global Shimura varieties and their quotients. We leave these generalizations to the interested reader.

2.7. **Organization.** In Section 3 we treat some preliminaries on nilpotent thickenings of adic spaces and schemes. In Section 4 we develop the theory of inscription in a general axiomatic setting. In Section 5 we then specialize to the specific contexts of interest to us in p-adic geometry. In Section 6 we study the inscribed moduli stacks of vector bundles and G-bundles on thickenings of relative Fargues-Fontaine curves (and other loci on relative Fargues-Fontaine curves). In Section 7 we establish our results on B^+_{dR}-affine Grassmannians, including Theorem A. In Section 8 we discuss modifications of bundles on relative thickened Farguess-Fontaine curves and related inscribed objects; we also define generalized Newton strata. A key technical ingredient in many of our later computations of tangent bundles is the torsor structure on the inscribed Hecke correspondence in Section 8.3. In Section 9 we study moduli of modifications and prove (generalizations of) Theorem B and Theorem C. In Section 10 we prove our results on cohomological smoothness as described in Section 2.6, including Theorem E. Finally, in Section 11 we study global Shimura varieties and prove Theorem D.

3. Adic spaces and schemes

In this section we develop some complements on adic spaces and schemes. In particular, we study properties of finite locally free nilpotent thickenings and smooth morphisms, and the relation between them. Everything we discuss is completely classical in the context of schemes, but a bit more delicate in the category of adic spaces.

To avoid issues of sheafiness, we will work exclusively with the category of strongly sheafy adic spaces as introduced in [9] (see Definition 3.2.1). This category includes the sousperfectoid spaces used in [7, IV.4], and thus Fargues-Fontaine curves, but also includes, e.g., smooth rigid analytic varieties over $\mathrm{Spa}(L)$ for any non-archimedean field L (note that, for L very large, $\mathrm{Spa}L$ may not itself be sousperfectoid; see [9, Remarks 7.7 and 7.8]). Another option that would accommodate both of these is the category of weakly sousperfectoid spaces of [9], but we will also want to allow locally free nilpotent thickenings (as defined in Section 3.4 below) as well as more exotic non-reduced spaces such as the canonical infinitesimal thickenings of perfectoid spaces over $\mathbb{Q}_p$. All of these non-reduced spaces are strongly sheafy but not weakly sousperfectoid.

Crucially, there is a reasonable notion of smooth morphisms between strongly sheafy adic spaces such that some of the results on smooth morphisms of sous-perfectoid adic spaces obtained in [7, IV.4] still hold with little modification to the proofs. In particular, enough structural properties hold for smooth morphisms in this context for us to describe their local structure around a section, define the relative tangent bundles of a smooth morphism, and relate relative tangent bundles to restrictions of scalars along simple square-zero thickenings.

3.1. **Conventions.** In this paper, all Huber rings are complete Tate rings. We write AdicSpc for the category of analytic adic spaces, i.e. the category of adic spaces locally of the form $\mathrm{Spa}(A, A^+)$ for (A, A^+) a sheafy Huber pair such that A is a complete Tate ring.

3.2. **Strongly sheafy adic spaces.**

Definition 3.2.1 (cf. Definition 4.1 of [9])**.** A Huber pair (A, A^+) is *strongly sheafy* if, for every $n \geq 0$, the Tate algebra

$$(A\langle t_1, \ldots, t_n\rangle, A^+\langle t_1, \ldots, t_n\rangle)$$

is sheafy. We say an adic space X is *strongly sheafy* if it can be covered by open affinoids $\mathrm{Spa}(A, A^+)$, for (A, A^+) a strongly sheafy Huber pair. We write SSAdicSpc $\subseteq$ AdicSpc for the full subcategory of strongly sheafy adic spaces.

Example 3.2.2.

(1) Any sous-perfectoid adic space is strongly sheafy. In particular, this applies to any $P \in \mathrm{Perf}$, the Fargues-Fontaine curve $X_{E,P}$ and its cover $Y_{E,P}$, and any untilt $P^\sharp$.
(2) Any strongly Noetherian adic space is strongly sheafy. In particular, this applies to rigid analytic varieties over non-archimedean fields.
(3) For $P^\sharp$ a perfectoid space over $\mathbb{Q}_p$, its canonical thickenings $P^\sharp_{(i)}$ are strongly sheafy.

By Corollary 3.4.5 below, any locally free nilpotent thickening of a strongly sheafy adic space is strongly sheafy.

3.3. **Vector bundles.** We write Vect for the fibered category over locally ringed spaces whose objects are pairs $(T, \mathcal{F})$ where T is a locally ringed space and $\mathcal{F}$ is a locally free of finite rank sheaf of $\mathcal{O}_T$-modules.

Given an affine scheme $\mathrm{Spec}\, A$, global sections give an equivalence between $\mathrm{Vect}(\mathrm{Spec}\, A)$ and the category of finite projective A-modules (see, e.g, [26, Tag 00NX]). Similarly, given an affinoid adic space $\mathrm{Spa}(A, A^+)$, global sections gives an equivalence between $\mathrm{Vect}(\mathrm{Spa}(A, A^+))$ and the category of finite projective A-modules by [16, Theorem 8.2.22].

The restriction of Vect to the category of schemes is an étale stack. Similarly, because strongly sheafy adic spaces are stably adic in the sense of [16, Definition 8.2.19], [16, Theorem 8.2.22] implies that the restriction of Vect to SSAdicSpc is an étale stack.

3.4. **Thickenings of adic spaces and schemes.**

Definition 3.4.1.

(1) A closed immersion $T \to T'$ of adic spaces is a nilpotent thickening if the ideal sheaf $\mathcal{I}_T$ of T in $\mathcal{O}_{T'}$ is locally nilpotent, i.e. after restriction to any quasi-compact open there is an n such that $\mathcal{I}_T^n = 0$. It is *square-zero* if $\mathcal{I}_T^2 = 0$.
(2) A nilpotent thickening of T/T is a morphism $T/T \to T'/T$ of adic spaces over T such that $T \to T'$ is a nilpotent thickening. Given such a nilpotent thickening, we obtain a splitting $\mathcal{O}_{T'} = \mathcal{O}_T \oplus \mathcal{I}_T$ in the category of sheaves of complete topological $\mathcal{O}_T$-modules. We say $T/T \to T'/T$ is locally free if
   (a) $\mathcal{O}_{T'}$, or equivalently $\mathcal{I}_T$, is locally free of finite rank over $\mathcal{O}_T$, and
   (b) For any open affinoid adic $\mathrm{Spa}(A, A^+) \subseteq T$, $\mathcal{O}_{T'}(\mathrm{Spa}(A, A^+))$, or equivalently $\mathcal{I}_T(\mathrm{Spa}(A, A^+)$, is equipped with its canonical topology as a finite projective A-module.
(3) For T an adic space, we say an $\mathcal{O}_T$-algebra $\mathcal{A}$ is augmented if it is equipped with a (necessarily surjective) map of $\mathcal{O}_T$-algebras $\mathcal{A} \to \mathcal{O}_T$, and nilpotent augmented if the kernel $\mathcal{I}$ of the augmentation is locally nilpotent. We say an augmented $\mathcal{O}_T$-algebra $\mathcal{A}$ is locally free if $\mathcal{I}$ or equivalently $\mathcal{A}$ is locally free of finite rank over $\mathcal{O}_T$.

Remark 3.4.2. Any thickening $T \to T'$ induces a homeomorphism $|T| = |T'|$ and for any point $t \in |T|$ with associated valuation v_t on $\mathcal{O}_{T,t}$, the induced valuation on $\mathcal{O}_{T',t}$ is given by composition of v_t with $\mathcal{O}_{T',t} \twoheadrightarrow \mathcal{O}_{T,t}$.

For any locally free nilpotent thickening $T/T \to T'/T$, it follows from the definitions that $\mathcal{O}_{T'}$ is a locally free nilpotent augmented $\mathcal{O}_T$-algebra. Conversely, given a locally free nilpotent augmented $\mathcal{O}_T$-algebra $\mathcal{A}$, we may construct a locally free nilpotent thickening $T/T \to \mathrm{Spa}_T\mathcal{A}$ as follows: we write $\mathrm{Spa}_T\mathcal{A}$ for the locally v-ringed space $(T, \mathcal{A})$, where for any open affinoid $\mathrm{Spa}(A, A^+) \subseteq T$, $\mathcal{A}(\mathrm{Spa}(A, A^+))$ is equipped with its canonical topology as a finite projective A-module and, for each $t \in T$, the valuation on $\mathcal{A}_t$ is pulled back from the valuation t on $\mathcal{O}_{T,t}$ along the surjection $\mathcal{A}_t \to \mathcal{O}_{T,t}$. When $T = \mathrm{Spa}(A, A^+)$ is affine, then one easily checks that $\mathrm{Spa}_T\mathcal{A} = \mathrm{Spa}(\mathcal{A}(T), A^+ \oplus \mathcal{I}(T))$, where $\mathcal{A}(T)$ is equipped with its canonical topology as a finite projective A-module. Equipped with the map $T \to \mathrm{Spa}_T\mathcal{A}$ coming from the augmentation and the structure map $\mathrm{Spa}_T\mathcal{A} \to T$ coming from the $\mathcal{O}_T$-algebra structure, $\mathrm{Spa}_T\mathcal{A}/T$ is a thickening of T/T.

Note that we can make analogous definitions with schemes, where everything is simpler as we do not need to keep track of the topology. The following is then immediate from the above discussion.

Proposition 3.4.3.

- *For T an adic space (resp. scheme), the functor*

$$(T/T \hookrightarrow T'/T) \mapsto (\mathcal{O}_{T'} \to \mathcal{O}_T)$$

is an equivalence between locally free nilpotent thickenings of T/T and locally free nilpotent augmented $\mathcal{O}_T$-algebras, with quasi-inverse

$$(\mathcal{A} \to \mathcal{O}_T) \mapsto (T/T \hookrightarrow \mathrm{Spa}_T \mathcal{A}/T) \text{ (resp. } (T/T \hookrightarrow \mathrm{Spec}_T \mathcal{A}/T)).$$

- *For T an adic space (resp. scheme), $f : V \to T$ a map of adic space (resp. schemes), and $\mathcal{A}$ a locally free augmented $\mathcal{O}_T$-algebra, $f^*\mathcal{A}$ is a locally free augmented $\mathcal{O}_V$-algebra, and*

  $$\mathrm{Spa}_V f^*\mathcal{A} = \mathrm{Spa}_T \mathcal{A} \times_T V \text{ (resp. } \mathrm{Spec}_V f^*\mathcal{A} = \mathrm{Spec}_T \mathcal{A} \times_T V).$$

  In particular, the category of locally free nilpotent thickenings is fibered over adic spaces (resp. schemes).

Corollary 3.4.4. *For any sheafy Huber pair (A, A^+), there is a natural equivalence between the category locally free thickenings of* $\mathrm{Spa}(A, A^+)/\mathrm{Spa}(A, A^+)$ *and locally free thickenings of* $\mathrm{Spec}\, A/\mathrm{Spec}\, A$.

Proof. We apply Proposition 3.4.3 along with the observation that $\mathrm{Vect}(\mathrm{Spa}(A, A^+))$ and $\mathrm{Vect}(\mathrm{Spec}\, A)$ are both equivalent, by global sections, to finite projective A-modules (see Section 3.3). □

Corollary 3.4.5. *If T is a strongly sheafy adic space, then for any locally free nilpotent thickening $T/T \to T'/T$, T' is a strongly sheafy adic space.*

Remark 3.4.6. If T is reduced, then for any finite locally free thickening $T/T \to T'/T$, the map $T \to T'$ is uniquely determined by $T' \to T$ as the inverse of the induced isomorphism $(T')^{\mathrm{red}} \to T$

Definition 3.4.7. If T is an adic space (resp. a scheme), we say a locally free nilpotent thickening $T/T \to T'/T$ is constant if it there a finite free nilpotent augmented $\mathbb{Z}$-algebra $A \to \mathbb{Z}$ such that T'/T is isomorphic to $\mathrm{Spa}_T(A \otimes_{\mathbb{Z}} \mathcal{O}_T)$ (resp. $\mathrm{Spec}_T(A \otimes_{\mathbb{Z}} \mathcal{O}_T)$).

For T an adic space or scheme, it follows from Proposition 3.4.3 that the category of square-zero locally free nilpotent thickening of T/T is equivalent to the category of locally free $\mathcal{O}_T$-modules of finite rank.

Definition 3.4.8. Given an adic space or scheme T and a locally free of finite rank $\mathcal{O}_T$-module $\mathcal{I}$, we write $T[\mathcal{I}]$ for the associated locally free square-zero thickening of T. As a locally ringed space it is $(|T|, \mathcal{O}_T \oplus \mathcal{I})$ where the multiplication is given by $(a, i)(a', i') = (aa', ai' + a'i)$.

Definition 3.4.9. Given an adic space (resp. scheme) T and a finite free $\mathbb{Z}$-module M, we write

$$T[M] := T[\mathcal{O}_T \otimes M] = \mathrm{Spa}_T(\mathbb{Z}[M] \otimes \mathcal{O}_T) \text{ (resp. } \mathrm{Spec}_T(\mathbb{Z}[M] \otimes \mathcal{O}_T[M]))$$

where here $\mathbb{Z}[M]$ is the nilpotent augmented finite free $\mathbb{Z}$-algebra $\mathbb{Z} \oplus M$ with multiplication given by $(a, m)(a', m') = (aa', am' + a'm)$.

From the discussion above we see a constant square-zero thickening of T/T can be equivalently defined as a thickening isomorphic to $T[M]/T$ for some finite free $\mathbb{Z}$-module M. The following is immediate.

Lemma 3.4.10. *For any adic space T or scheme T, $M \mapsto T[M]/T$ defines a contravariant functor from finite free $\mathbb{Z}$-modules to constant square-zero thickenings of T/T such that*

$$T[M_1 \times M_2] = T[M_1] \sqcup_T T[M_2] \tag{3.4.10.1}$$

where the inclusion maps $T[M_i] \to T[M_1 \times M_2]$ are induced by the projections $M_1 \times M_2 \to M_i$.

Definition 3.4.11. For any subcategory $\mathcal{C}$ of adic spaces or schemes, we write $\mathcal{C}^{\mathrm{lf}}$ for the category of locally free nilpotent thickenings of objects in $\mathcal{C}$, $\mathcal{C}^{\mathrm{cn}}$ for the subcategory of $\mathcal{C}^{\mathrm{lf}}$ consisting of constant nilpotent thickenings of objects in $\mathcal{C}$, and $\mathcal{C}^{\epsilon}$ for the subcategory of $\mathcal{C}^{\mathrm{cn}}$ consisting of constant square-zero thickenings of objects in $\mathcal{C}$. In each case $\mathcal{C}^{\bullet}$ is naturally fibered over $\mathcal{C}$.

3.5. **Smooth morphisms.** The results on smooth morphisms of sous-perfectoid adic spaces of [7, IV.4] up through [7, Lemma IV.14] go through essentially as written in the more general setting of strongly sheafy adic spaces. In this section we give just the specific definitions and statements we will need.

Remark 3.5.1. The remaining statements in [7, IV.4] depend on Lemma [7, Lemma IV.16], whose proof in loc. cit. uses the sous-perfectoid condition to reduce to the case of a perfectoid X. We do not attempt to extend these results here as, for our purposes, the analog of [7, Lemma IV.14] saying that any section factors through a ball is the key structural result we need going forward. The generalization to strongly sheafy adic spaces is Proposition 3.5.6, and we deduce from it a useful corollary about infinitesimal neighborhoods of sections in Corollary 3.5.7.

Definition 3.5.2. (cf. [7, Definition IV.4.9], [9, Definition 5.11]). A morphism $f : Y \to X$ of strongly sheafy adic spaces is

(1) étale if, locally on X and Y, it can be written as an open immersion followed by a finite étale map, and
(2) smooth if there is a cover of Y by open sets V such that $f|_V$ can be written as a composition of an étale map $V \to \mathbb{B}^d_X$ followed by projection to X.

Lemma 3.5.3 (cf. Proposition IV.4.10-(iii) of [7])**.** *Let $Y \to X$ be a smooth map of strongly sheafy adic spaces, and let $X' \to X$ be an arbitrary map of strongly sheafy adic spaces. Then, $Y' := Y \times_X X'$ is a strongly sheafy adic space and $Y' \to X'$ is a smooth map of strongly sheafy adic spaces. Thus, the category of smooth morphisms of strongly sheafy adic spaces is a fibered category over strongly sheafy adic spaces.*

Proof. This follows since balls over, rational localizations of, and finite étale covers of a strongly sheafy $\mathrm{Spa}(A, A^+)$ are all strongly sheafy. □

Definition 3.5.4 (cf. Definition IV.4.11 of [7])**.** For $f : Y \to X$ a smooth map of strongly sheafy adic spaces, the sheaf of relative differentials $\Omega_{Y/X}$ on Y is $\mathcal{I}/\mathcal{I}^2$ for $\mathcal{I}$ the ideal sheaf of the diagonal $Y \hookrightarrow Y \times_X Y$.

Lemma 3.5.5. *For $f : Y \to X$ a smooth map of strongly sheafy adic spaces, $\Omega_{Y/X}$ is locally free of finite rank over $\mathcal{O}_Y$. In fact it is free of rank d on any open V as in the definition of a smooth map such that $f|_V$ factors through an étale map $V \to \mathbb{B}^d_X$. For any map of strongly sheafy adic spaces $g : X' \to X$, $g^*\Omega_{Y/X} = \Omega_{Y\times_X X'/X'}$.*

Proof. The following is based on the proof of [7, Proposition IV.4.12], but we have modified the structure of the argument to make it more clear (to us). Working locally, we may assume $X = \mathrm{Spa}(A, A^+)$, Y is quasi-compact, and $Y \to X$ factors through an étale map $Y \to \mathbb{B}^d_X =: Y'$. The diagonal $Y \to Y \times_X Y$ then factors as

$$Y \to Y \times_{Y'} Y = (Y \times_X Y) \times_{Y' \times_X Y'} Y' \to Y \times_X Y.$$

The first map, as the diagonal of an étale map, is an open immersion. It follows that $\mathcal{I}_Y/\mathcal{I}_Y^2$ is the restriction of $\mathcal{I}_{Y\times_{Y'}Y}/\mathcal{I}^2_{Y\times_{Y'}Y}$ to Y.

We thus analyze this ideal sheaf. By a change of coordinates, we may rewrite $Y' \times_X Y'$ as $\mathbb{B}^{2d}_X$, such that the diagonal of Y' over X is the inclusion $\mathbb{B}^d_X \hookrightarrow \mathbb{B}^{2d}_X$ corresponding to setting the first d coordinates to zero. In these coordinates, we write $Z \to \mathbb{B}^{2d}_X$ for the étale map $Y \times_X Y \to Y' \times_X Y'$ and $Z_0 \to \mathbb{B}^d_X$ for its restriction to the diagonal of Y' over X obtained by setting the first d coordinates to zero, i.e. for $Y \times_{Y'} Y \to Y'$. We are thus interested in the ideal sheaf of Z_0 in Z. By spreading out of the étale site [22, Lemma 15.6, Lemma 12.17] we find that, for $n \gg 0$, $Z|_{p^n\mathbb{B}^d_X \times_X \mathbb{B}^d_X}$ is isomorphic to $p^n\mathbb{B}^d_X \times_X Z_0$ over $p^n\mathbb{B}^d_X \times_X \mathbb{B}^d_X$. Covering Z_0 by strongly sheafy affinoids $\mathrm{Spa}(R, R^+)$, the corresponding opens $p^n\mathbb{B}^d_X \times_X \mathrm{Spa}(R, R^+)$ are of the form $\mathrm{Spa}(R\langle t_1, \dots, t_d\rangle, R^+\langle t_1, \dots, t_d\rangle)$ and the ideal sheaf is $\mathcal{I}_Z = (t_1, \dots, t_d)$. It follows that on this affinoid $\mathcal{I}_Z/\mathcal{I}_Z^2$ is free of rank d with basis the classes of $t_1, \dots, t_d$, and that concludes the proof of the local freeness and computation of the rank. Using these local charts, the claim about the base change is also immediate.

□

Proposition 3.5.6. *(cf. [7, Lemma IV.4.14]). If $f : Y \to X$ is a smooth map of strongly sheafy adic spaces and $s : X \to Y$ is a section, then there is a cover of X by open subsets U such that $s|_U$ factors through a neighborhood of $Y|_U$ isomorphic to $\mathbb{B}^d_U$ (as adic spaces over U).*

Proof. The proof from [7, Lemma IV.4.14] applies with no change. □

Corollary 3.5.7. *If $f : Y \to X$ is a smooth map of strongly sheafy adic spaces and $s : X \to Y$ is a section then, for any $n \geq 0$, the nth infinitesimal neighborhood $s_{(n)}$ of $s(X)$ in Y is a locally free nilpotent thickening of X/X, and there is a natural isomorphism $s_{(1)} = X[s^*\Omega_{Y/X}]$.*

Proof. For the first part, note that over any U as in Proposition 3.5.6, we can translate so that s is the zero section. Then, over U, $s_{(n)}$ is the thickening associated to $\mathcal{O}_U[t_1, \dots, t_d]/(t_1, \dots, t_d)^{n+1}$ by Proposition 3.4.3.

For the second part, to see that there is a natural isomorphism $s_{(1)} = X[s^*\Omega_{Y/X}]$, it suffices to construct a natural map $s^*\Omega_{Y/X} \to \mathcal{I}_{s(X)}/\mathcal{I}^2_{s(X)}$ and then verify it is an isomorphism in these local coordinates. Now, for $\Delta : Y \to Y \times_X Y$ the diagonal, we have

$$s^*\Omega_{Y/X} = s^*\Delta^*\mathcal{I}_\Delta = (\Delta \circ s)^*\mathcal{I}_\Delta$$

On the other hand, we can also write

$$\Delta \circ s = ((s \circ f) \times \mathrm{Id}) \circ s.$$

The restriction of $((s \circ f) \times \mathrm{Id})^*\mathcal{O}_{Y \times_X Y} \to \mathcal{O}_Y$ to $((s \circ f) \times \mathrm{Id})^*\mathcal{I}_\Delta$ factors through $\mathcal{I}_{s(X)}$. Thus we obtain a natural map

$$s^*\Omega_{Y/X} = (((s \circ f) \times \mathrm{Id}) \circ s)^*\mathcal{I}_\Delta \to s^*\mathcal{I}_{s(X)} = \mathcal{I}_{s(X)}/\mathcal{I}^2_{s(X)}.$$

Using the local coordinates as above for the zero section in $\mathbb{B}^d_U$, it is elementary to check that this is an isomorphism. □

3.6. **Tangent bundles and restriction of scalars.** We now discuss the relation between tangent bundles and restriction of scalars for smooth morphisms. We first recall the theory for schemes.

First, we recall that for any scheme X and quasi-coherent sheaf $\mathcal{F}/X$, we can form $\mathbb{V}(\mathcal{F}) := \mathrm{Spec}_X \mathrm{Sym}^\bullet \mathcal{F}$, a scheme over X representing

$$(T/X) \mapsto \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{F}_T, \mathcal{O}_T).$$

This construction is naturally a contravariant functor from quasi-coherent sheafs over X to schemes over X.

Now, suppose Y/X is a morphism of schemes, and $\mathcal{I}$ is a locally free of finite rank sheaf of $\mathcal{O}_X$-modules on X. Then, we may consider the restriction of scalars $R_{X[\mathcal{I}]/X}(Y[\mathcal{I}_Y]/X[\mathcal{I}])$. This is the functor on schemes over X sending T/X to

$$\begin{aligned} \mathrm{Hom}_{X[\mathcal{I}]}(T \times_X X[\mathcal{I}], Y[\mathcal{I}_Y]) &= \mathrm{Hom}_{X[\mathcal{I}]}(T \times_X X[\mathcal{I}], Y \times_X X[\mathcal{I}]) \\ &= \mathrm{Hom}_X(T \times_X X[\mathcal{I}], Y). \end{aligned}$$

Pull-back along the closed immersion $X \hookrightarrow X[\mathcal{I}]$ equips $R_{X[\mathcal{I}]/X}(Y[\mathcal{I}_Y]/X[\mathcal{I}])$ with a structure map to Y. The following shows it is represented by a natural scheme over Y.

Proposition 3.6.1. *If $f : Y \to X$ is a morphism of schemes, there is a natural identification of functors on schemes over Y*

$$R_{X[\mathcal{I}]/X}(Y[\mathcal{I}_Y]/X[\mathcal{I}]) = \mathbb{V}(\Omega_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{I}_Y^*).$$

Proof. To give a map from $T \times_X X[I]/X$ to Y/X inducing a fixed map $g : T \to Y$ is the same as to give a map of $f^{-1}\mathcal{O}_X$-algebras augmented to $\mathcal{O}_Y$

$$\mathcal{O}_Y \to g_*\mathcal{O}_T \oplus (g_*\mathcal{I}_T).$$

This is equivalent to an $f^{-1}\mathcal{O}_X$-linear derivation $\mathcal{O}_Y \to g_*\mathcal{I}_T$, or equivalently an element of

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_Y}(\Omega_{Y/X}, g_*\mathcal{I}_T) &= \mathrm{Hom}_{\mathcal{O}_T}(g^*\Omega_{Y/X}, \mathcal{I}_T)\\ &= \mathrm{Hom}_{\mathcal{O}_T}(g^*\Omega_Y, g^*\mathcal{I}_Y)\\ &= \mathrm{Hom}_{\mathcal{O}_T}(g^*(\Omega_Y \otimes_{\mathcal{O}_Y} \mathcal{I}_Y^*), \mathcal{O}_T)\\ &= \mathrm{Hom}_Y(T/Y, \mathbb{V}(\Omega_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{I}_Y^*)).\end{aligned}$$

□

We now want to imitate this with strongly sheafy adic spaces. In this case, the functor $\mathbb{V}$ is defined only for $\mathcal{F}$ locally free (in which case the sheaf of sections of $\mathbb{V}(\mathcal{F})$ is $\mathcal{F}^*$). This is not an issue, since for strongly sheafy adic spaces we have anyway only defined $\Omega_{Y/X}$ for Y/X smooth.

Another issue that does arise, however, is that even in the smooth case where we have defined $\Omega_{Y/X}$, we have not actually established any universality property for the natural derivation $\mathcal{O}_Y \to \Omega_{Y/X}$. In fact, the notion of a universal continuous derivation is a bit subtle and may be best understood in a more general context. We avoid this by modifying the structure of the proof to work around the necessity of establishing such a property.

Proposition 3.6.2. *If Y/X is a smooth morphism of strongly sheafy adic spaces and $\mathcal{I}$ is a locally free of finite rank $\mathcal{O}_X$-module, then there is a natural identification of functors on strongly sheafy adic spaces over Y*

$$R_{X[\mathcal{I}]/X}(Y[\mathcal{I}_Y]/X[\mathcal{I}]) = \mathbb{V}(\Omega_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{I}_Y^*).$$

Proof. To give a map from $T \times_X X[\mathcal{I}]/X$ to Y/X inducing a fixed map $g : T \to Y$ is the same as to give an extension of the section $s = g \times \mathrm{Id} : T \to Y \times_X T$ to a section

$$\tilde{s} : T[\mathcal{I}_T]/T \to Y \times_X T/T.$$

Such a map must factor through the first infinitesimal neighborhood $s_{(1)}$ of s in $Y \times_X T$. Thus, by Corollary 3.5.7, to give $\tilde{s}$ is the same as to give a map of locally free square-zero thickenings

$$T[\mathcal{I}_T]/T \to T[s^*\Omega_{Y \times_X T/T}]/T = T[g^*\Omega_{Y/X}]/T.$$

But by Proposition 3.4.3, this is the same as to give a map of $\mathcal{O}_T$-modules $g^*\Omega_{Y/X} \to \mathcal{I}_T$. This yields the desired equality

$$\begin{aligned}\mathrm{Hom}(g^*\Omega_{Y/X}, \mathcal{I}_T) &= \mathrm{Hom}(g^*(\Omega_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{I}_Y^*), \mathcal{O}_T)\\ &= \mathrm{Hom}_Y(T/Y, \mathbb{V}(\Omega_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{I}_Y^*))\end{aligned}$$

□

Example 3.6.3. When $\mathcal{I} = \mathcal{O}_X \cdot \epsilon$ in Proposition 3.6.1 or Proposition 3.6.2, writing $T_{Y/X} := \mathbb{V}(\Omega_{Y/X})$ for the (geometric) tangent bundle of Y over X, we find

$$R_{X[\epsilon]/X]}(Y[\epsilon]/X[\epsilon]) = T_{Y/X}.$$

4. Inscription

In this section we introduce the notion of inscription and the construction of tangent bundles for inscribed objects. The general setup starts with a pair $(\mathcal{C}, B)$, where $\mathcal{C}$ is a category and B is a functor from $\mathcal{C}$ to schemes or strongly sheafy adic spaces. In this setup, we define a new category B^{lf} whose objects are pairs consisting of an object of $\mathcal{C}$ and a locally free thickening of the associated scheme or strongly sheafy adic space. An inscribed fibered category will then be a fibered category over B^{lf} that transforms certain simple coproducts into products. These objects have natural tangent bundles, which are again inscribed fibered categories, and for an inscribed presheaf the tangent bundle has moreover a natural module structure. After developing the basic language, in Section 4.7 we explain a simple theory of inscribed groups and their actions, and in Section 4.8 we explain a moduli of sections construction of inscribed presheaves that will play a key role in the remainder of the work. In section Section 4.9 we discuss how these results extend to a more general notion where the test objects form only a natural subcategory of B^{lf}.

The key example for us is when $\mathcal{C}$ is a category of affinoid perfectoid spaces and the functor B is the Fargues-Fontaine curve. However, we will also find ourselves using variants where B is another natural functor arising in p-adic Hodge theory, such as the canonical deformation of a perfectoid space in characteristic zero. Since the basic properties are completely independent of any constructions in p-adic Hodge theory, we hope it will be clearer to develop them without mentioning the specific situation. The specific inscribed contexts we are interested in will be described in detail in Section 5, along with the results specific to those setups.

4.1. Inscribed fibered categories.

We adopt the terminology of [26, Tag 0011] in our discussion of fibered categories.

Let $\mathcal{C}$ be a category, let Spaces be either the category of schemes or strongly sheafy adic spaces and let $B : \mathcal{C} \to \text{Spaces}$ be a covariant functor.

Definition 4.1.1. We write B^{lf} for the fibered category over $\mathcal{C}$ whose objects are pairs $(o, \mathcal{B}/B(o))$ such that $o \in \mathcal{C}$ and $\mathcal{B}/B(o)$ is a locally free nilpotent thickening of $B(o)/B(o)$ (see Definition 3.4.1). The morphisms

$$\mathrm{Hom}_{B^{\mathrm{lf}}}\big((o, \mathcal{B}/B(o)), (o', \mathcal{B}'/B(o'))\big)$$

are given by the set of pairs consisting of a morphism $o \to o'$ and a morphism $\mathcal{B} \to \mathcal{B}'$ covering the induced map $B(o) \to B(o')$. In what follows, we will write an object of B^{lf} simply as $\mathcal{B}$ or $\mathcal{B}/B(o)$ when it will cause no confusion.

Example 4.1.2. If $\mathcal{C} = \{*\}$ and $B(*) = \operatorname{Spec} k$ (resp. $\operatorname{Spa}(k, k^+)$) for k a field (resp. non-archimedean field), then B^{lf} is equivalent to the opposite category of nilpotent artininian local k-algebras with residue field k.

Note that, given an object $\mathcal{B}_0/B(o) \in B^{\mathrm{lf}}$, and locally free nilpotent thickenings $\mathcal{B}_i/\mathcal{B}_0$ of $\mathcal{B}_0/\mathcal{B}_0$, $i = 1, 2$, the push-out $\mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2$ is naturally a locally free nilpotent thickening of $B(o)/B(o)$ thus gives an object of B^{lf}. At the level of augmented $\mathcal{O}_{B(o)}$ algebras as in Proposition 3.4.3, this push-out corresponds to the fiber product $\mathcal{O}_{\mathcal{B}_1} \times_{\mathcal{O}_{\mathcal{B}_0}} \mathcal{O}_{\mathcal{B}_2}$.

Definition 4.1.3 (Inscribed fibered categories)**.** A fibered category $\mathcal{S}$ over B^{lf} is *inscribed* if, for any $\mathcal{B}_0 \in B^{\mathrm{lf}}$ and any pair of locally free nilpotent thickenings $\mathcal{B}_i/\mathcal{B}_0$, $i = 1,2$, the functor

$$\mathcal{S}(\mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2) \to \mathcal{S}(\mathcal{B}_1) \times_{\mathcal{S}(\mathcal{B}_0)} \mathcal{S}(\mathcal{B}_2) \tag{4.1.3.1}$$

induced by pullback along $\mathcal{B}_i \hookrightarrow \mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2$ is an equivalence (note that this does not depend on the choice of pullbacks for $\mathcal{S}$).

Example 4.1.4. We can view a presheaf $\mathcal{S}$ on B^{lf} as a discrete fibered category, i.e. a category fibered in sets over B^{lf}. Then $\mathcal{S}$ is inscribed if and only for any $\mathcal{B}_i/\mathcal{B}_0$ as above, Eq. (4.1.3.1) is a bijection of sets.

Definition 4.1.5 (Underlying fibered category and trivial inscription)**.**

(1) Pullback along $\mathcal{C} \to B^{\mathrm{lf}}$, $o \mapsto B(o)/B(o)$ is a functor from the (2,1)-category of fibered categories over $\mathcal{B}$ to the (2,1)-category of fibered categories over $\mathcal{C}$, which we write as $\mathcal{S} \mapsto \mathcal{S}_0$. We refer to $\mathcal{S}_0$ as the underlying fibered category of $\mathcal{S}$; if $\mathcal{S}$ is inscribed, we refer to $\mathcal{S}$ as an inscription on $\mathcal{S}_0$.
(2) Pullback along $B^{\mathrm{lf}} \to \mathcal{C}$, $\mathcal{B}/B(o) \mapsto B(o)$, is a functor from the (2,1)-category of fibered categories over $\mathcal{C}$ to the (2,1)-category of fibered categories over B^{lf}, which we write as $S \mapsto S^{\mathrm{triv}}$. We refer to S^{triv} as the trivial inscription on S.

The name trivial inscription is justified by the immediate

Lemma 4.1.6. *For S a fibered category over $\mathcal{C}$, S^{triv} is inscribed.*

The composition of $\mathcal{C} \to B^{\mathrm{lf}} \to \mathcal{C}$ is the identity functor. We thus obtain a natural isomorphism of functors from fibred categories on $\mathcal{C}$ to fibered categories on $\mathcal{C}$, $(\square^{\mathrm{triv}})_0 = \mathrm{Id}$. We also have a natural transformations $\mathrm{Id} \to (\square_0)^{\mathrm{triv}} \to \mathrm{Id}$ of functors from fibered categories on B^{lf} to fibered categories on B^{lf} via the natural maps $\mathcal{S}(\mathcal{B}/B(o)) \to \mathcal{S}(B(o)/B(o)) \to \mathcal{S}(\mathcal{B}/B(o))$ induced by pullbacks along $\mathcal{B}/B(o) \to B(o)/B(o) \to \mathcal{B}/B(o))$.

The induced functors

$$\mathrm{Hom}(S^{\mathrm{triv}}, \mathcal{S}) \to \mathrm{Hom}(S, \mathcal{S}_0) \text{ and } \mathrm{Hom}(\mathcal{S}_0, S) \to \mathrm{Hom}(\mathcal{S}, S^{\mathrm{triv}})$$

are equivalences, i.e. trivial inscription and the underlying fibered category are ambidextrously adjoint. In particular, since $(\square^{\mathrm{triv}})_0 = \mathrm{Id}$, we find $\square^{\mathrm{triv}}$ is fully faithful. Because of this, we will often drop the superscript triv and simply treat fibred categories over $\mathcal{C}$ as trivially inscribed fibered categories over B^{lf} when it will cause no serious confusion.

We also have the following useful permanence property under limits:

Lemma 4.1.7. *Inscribed fibered categories are preserved under 2-limits.*

Proof. For $\mathcal{S}_i$, $j \in J$ a diagram of inscribed fibered categories and $\mathcal{B}_i/\mathcal{B}_0$ locally free thickenings of $\mathcal{B}_0$ as in Definition 4.1.3 we have

$$\begin{aligned}(\lim \mathcal{S}_j)(\mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2) &= \lim(\mathcal{S}_j(\mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2)) \\ &= \lim(\mathcal{S}_j(\mathcal{B}_1) \times_{\mathcal{S}_j(\mathcal{B}_0)} \mathcal{S}_j(\mathcal{B}_2)) \\ &= (\lim \mathcal{S}_j)(\mathcal{B}_1) \times_{(\lim \mathcal{S}_j)(\mathcal{B}_0)} (\lim \mathcal{S}_j)(\mathcal{B}_2).\end{aligned}$$

□

4.2. $\mathbb{B}$-modules.

Definition 4.2.1. Let $\mathbb{B}$ be the presheaf of rings on B^{lf} defined by

$$\mathbb{B}(\mathcal{B}) = H^0(\mathcal{B}, \mathcal{O}_{\mathcal{B}}).$$

Proposition 4.2.2. *$\mathbb{B}$ is an inscribed presheaf.*

Proof. To see that $\mathbb{B}$ is inscribed, it suffices to note that

$$\mathbb{B}(\mathcal{B}/B(o)) = \mathrm{Hom}_{B(o)}(\mathcal{B}, \mathbb{A}^1_{B(o)})$$

so that it follows from the universal property of the coproduct. □

Remark 4.2.3. In Section 4.8 we will see that much more general moduli of sections constructions also give rise to inscribed presheaves.

A $\mathbb{B}$-module over an inscribed presheaf $\mathcal{S}$ is a morphism of presheaves $\mathcal{V} \to \mathcal{S}$ equipped with a zero section $_{\mathcal{V}} : \mathcal{S} \to \mathcal{V}$, a commutative group law $\mathcal{V} \times_{\mathcal{S}} \mathcal{V} \to \mathcal{V}$, and an action map $\mathbb{B} \times \mathcal{V} \to \mathcal{V}$ satisfying the usual properties. A $\mathbb{B}$-module $\mathcal{V}$ over $\mathcal{S}$ is inscribed if it is inscribed as a presheaf on B^{lf}.

4.3. Tangent bundles.

There is a natural functor $B^{\mathrm{lf}} \to B^{\mathrm{lf}}$ of fibered categories over $\mathcal{C}$ sending $\mathcal{B}$ to $\mathcal{B}[\epsilon]$. For $\mathcal{S}$ an inscribed fibered category, we write $T_{\mathcal{S}}$ for the pullback of $\mathcal{S}$ along this functor.

Lemma 4.3.1. *For $\mathcal{S}$ an inscribed fibered category, $T_{\mathcal{S}}$ is an inscribed fibered category.*

Proof. Using the inscribed property of $\mathcal{S}$, we compute

$$\begin{aligned}(T_{\mathcal{S}})(\mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2) &= \mathcal{S}((\mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2)[\epsilon]) \\ &\cong \mathcal{S}(\mathcal{B}_1[\epsilon] \sqcup_{\mathcal{B}_0[\epsilon]} \mathcal{B}_2[\epsilon]) \\ &\cong \mathcal{S}(\mathcal{B}_1[\epsilon]) \times_{\mathcal{S}(\mathcal{B}_0[\epsilon])} \mathcal{S}(\mathcal{B}_2[\epsilon]) \\ &\cong T_{\mathcal{S}}(\mathcal{B}_1) \times_{T_{\mathcal{S}}(\mathcal{B}_0)} T_{\mathcal{S}}(\mathcal{B}_2).\end{aligned}$$

□

The canonical map $\mathcal{B} \to \mathcal{B}[\epsilon]$ induces a morphism $T_{\mathcal{S}} \to \mathcal{S}$. When $\mathcal{S}$ is an inscribed presheaf, so is $T_{\mathcal{S}}$, and we claim this morphism can be upgraded with the natural structure of a $\mathbb{B}$-module over $\mathcal{S}$. Indeed, we will show it obtains a 0-section $0_{T_{\mathcal{S}}} : \mathcal{S} \to T_{\mathcal{S}}$ by pullback along the structure maps

$\mathcal{B}[\epsilon] \to \mathcal{B}$, an action $a_{T_{\mathcal{S}}} : \mathbb{B} \times \mathcal{V} \to \mathcal{V}$ via the natural identification $\mathbb{B}(\mathcal{B}) = \mathrm{End}_{\mathcal{B}}(\mathcal{B}[\epsilon])$, and an abelian group structure $+_{T_{\mathcal{S}}} : T_{\mathcal{S}} \times_{\mathcal{S}} T_{\mathcal{S}}$ by

$$\mathcal{S}(\mathcal{B}[\epsilon]) \times_{\mathcal{S}(\mathcal{B})} \mathcal{S}(\mathcal{B}[\epsilon]) \to \mathcal{S}(\mathcal{B}[\epsilon] \sqcup_{\mathcal{B}} \mathcal{B}[\epsilon]) \to \mathcal{S}(\mathcal{B}[\epsilon])$$

where the first map is obtained by inverting the bijection coming from the inscribed property and the second map is pullback along the composition of $\mathcal{B}[\epsilon] \sqcup_{\mathcal{B}} \mathcal{B}[\epsilon] = \mathcal{B}[\epsilon_1, \epsilon_2]$ and the map $\epsilon \mapsto \epsilon_1 + \epsilon_2$.

Proposition 4.3.2. *The assignment*

$$\mathcal{S} \mapsto (T_{\mathcal{S}}/\mathcal{S}, 0_{T_{\mathcal{S}}}, +_{T_{\mathcal{S}}}, a_{T_{\mathcal{S}}})$$

is a functor from the (1-)category of inscribed presheaves to the (1-)category of inscribed presheaves equipped with an inscribed $\mathbb{B}$*-module.*

We will prove the proposition using the following structure: we write $\mathcal{B}^*\mathrm{Vect}$ for the category of pairs $(\mathcal{B}, \mathcal{I})$ consisting of a $\mathcal{B}$ in B^{lf} and a locally free of finite rank $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{I}$. There is a functor $F : (\mathcal{B}^*\mathrm{Vect}) \to B^{\mathrm{lf}}$ given by $(\mathcal{B}, \mathcal{I}) \mapsto \mathcal{B}[\mathcal{I}]$. For $\mathcal{S}$ an inscribed fibered category, we can thus consider $F^*\mathcal{S}$. Then, for example, $T_{\mathcal{S}}$ is the pullback $(\mathcal{O}_{\mathcal{B}} \cdot \epsilon \times \mathrm{Id})^* F^*\mathcal{S}$, and $0_{T_{\mathcal{S}}}$ is obtained from $\mathcal{B} \mapsto (\mathcal{O}_{\mathcal{B}} \cdot \epsilon \xrightarrow{0} 0)$. The structures defining the group and $\mathbb{B}$-module structure also only depend on the restriction to the full fibered subcategory $\mathcal{O}_{\mathcal{B}}^{\oplus}$ whose objects are pairs $(\mathcal{B}, \mathcal{V})$ where $\mathcal{V}$ is a finite free module over $\mathcal{O}_{\mathcal{B}}$. We identify this with the category $\mathbb{B}^{\oplus}$ whose objects are pairs $(\mathcal{B}, M)$, where M is a finite free module over $\mathbb{B}(\mathcal{B})$.

The result will then follow from the following more general statement: Let $\mathcal{A}$ be a category, and let $\mathbb{A}$ be a presheaf of rings on $\mathcal{A}$, and let $\mathbb{A}^{\oplus}$, the fibered category of whose objects are pairs (A, M) for $A \in \mathbb{A}$ and M a finite free $\mathbb{A}(A)$-module. Note that the fiber of $\mathbb{A}^{\oplus}$ over A is the opposite category of finite free $\mathbb{A}(A)$-modules, so the restriction of any presheaf $\mathcal{F}$ to this fiber can be viewed as a covariant functor $\mathcal{F}_A$ from finite free $\mathbb{A}(A)$-modules to sets. We say a presheaf $\mathcal{F}$ on $\mathbb{A}^{\oplus}$ is product-preserving if each of its restrictions $\mathcal{F}_A$ preserve products over the final object, i.e. if the natural map $\mathcal{F}_A(M_1 \times M_2) \to \mathcal{F}_A(M_1) \times_{\mathcal{F}_A(0)} \mathcal{F}_A(M_2)$ is a bijection for any M_1, M_2. Given a product preserving $\mathcal{F}$, we write $\mathcal{F}_i$ for the presheaf on $\mathcal{A}$ sending A to $\mathcal{F}_i(\mathbb{A}(A)^i)$. Then, there are natural maps

$$\mathcal{F}_1 \to \mathcal{F}_0, 0_{\mathcal{F}} : \mathcal{F}_0 \to \mathcal{F}_1, m_{\mathcal{F}} : \mathcal{F}_1 \times_{\mathcal{F}_0} \mathcal{F}_1, \text{ and } a_{\mathcal{F}} : \mathbb{A} \times \mathcal{F}_1 \to \mathcal{F}_1$$

defined on the fiber over A as follows, writing $\mathbb{A}(A) =: R$,

(1) We define the structure map $\mathcal{F}_1(A) = \mathcal{F}_A(R) \to \mathcal{F}_A(0) = \mathcal{F}_0(A)$ by applying $\mathcal{F}_A$ to the final map $R \to 0$.
(2) We define $0_{\mathcal{F},A} : \mathcal{F}_0(A) = \mathcal{F}_A(0) \to \mathcal{F}_A(R) = \mathcal{F}_1(A)$ by applying $\mathcal{F}_A$ to the initial map $0 \to R$.
(3) We define $m_{\mathcal{F},A}$ by composing the inverse of the bijection

$$\mathcal{F}_A(R^2) \xrightarrow{\mathcal{F}_A(\begin{bmatrix}1 & 0\end{bmatrix}) \times \mathcal{F}_A(\begin{bmatrix}0 & 1\end{bmatrix})} \mathcal{F}(R) \times_{\mathcal{F}(0)} \mathcal{F}(R) = (\mathcal{F}_1 \times_{\mathcal{F}_0} \mathcal{F}_1)(A)$$

with

$$\mathcal{F}_A(R^2) \xrightarrow{\mathcal{F}_A(\begin{bmatrix}1 & 1\end{bmatrix})} \mathcal{F}_A(R) = \mathcal{F}_1(A).$$

(4) We define the action under $a_{\mathcal{F},A}$ of $r \in \mathbb{A}(A) = R$ on $\mathcal{F}_1(A) = \mathcal{F}_A(R)$ to be given by $\mathcal{F}_A([r])$, where $[r]$ is the 1×1 matrix viewed as the homomorphism from R to R given by left multiplication by r.

Lemma 4.3.3. *The assignment $\mathcal{F} \mapsto (\mathcal{F}_1/\mathcal{F}_0, 0_{\mathcal{F}}, m_{\mathcal{F}}, a_{\mathcal{F}})$ is an equivalence of categories between product preserving presheaves on $\mathbb{A}^{\oplus}$ and the category of pairs consisting of a presheaf $\mathcal{F}_0$ on $\mathcal{A}$ and an $\mathbb{A}$-module $\mathcal{F}_1/\mathcal{F}_0$.*

Proof. That the data defines an $\mathbb{A}$-module follows by arguing, for each $A \in \mathcal{A}$, fiberwise over $\mathcal{F}_0(A)$ using the usual result in the deformation theory of functors (see, e.g. [26, Tag 06I6]). In fact, one can just rewrite the arguments in this setting: for example, the commutativity of the group law follows from the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{F}_A(R) \times \mathcal{F}_A(R) & & \xrightarrow{(s_1,s_2)\mapsto(s_2,s_1)} & & \mathcal{F}_A(R) \times \mathcal{F}_A(R) \\
\uparrow{\scriptstyle (\mathcal{F}_A([1\,0]),\mathcal{F}_A([0\,1]))} & & & & \uparrow{\scriptstyle (\mathcal{F}_A([1\,0]),\mathcal{F}_A([0\,1]))} \\
\mathcal{F}_A(R^2) & & \xrightarrow{\mathcal{F}_A(\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right])} & & \mathcal{F}_A(R^2) \\
& \searrow{\scriptstyle \mathcal{F}_A([1\,1])} & & \swarrow{\scriptstyle \mathcal{F}_A([1\,1])} & \\
& & \mathcal{F}_A(R) & &
\end{array}$$

and similar diagrams establish the other R-module properties.

An inverse functor can be constructed by sending $\mathcal{F}_1/\mathcal{F}_0$ to the presheaf $\mathcal{F}$ sending (A, M) to the set of pairs (s, m) where $s \in \mathcal{F}_0(A)$ and $m \in M \otimes_{\mathbb{A}(A)} (\mathcal{F}_1(A) \times_{\mathcal{F}_0(A)} s)$; we omit the verification that this is an inverse. □

Proof of Proposition 4.3.2. The functor in Proposition 4.3.2 is given by first pulling back along $\mathbb{B}^{\oplus} \to \mathcal{B}^*\mathrm{Vect} \to B^{\mathrm{lf}}$ and then applying the functor of Lemma 4.3.3. That the pullback to $\mathbb{B}^{\oplus}$ is product preserving follows from the inscribed property. This shows we obtain a functor from inscribed presheaves to inscribed presheaves equipped with a $\mathbb{B}$-module, as claimed. That this $\mathbb{B}$-module is also inscribed follows from Lemma 4.3.1. □

Remark 4.3.4. For a general inscribed fibered category $\mathcal{S}$, it would thus be natural to try to replace the consideration of the tangent bundle with its group structure that we used for inscribed v-sheaves with the consideration of the pullback of $\mathcal{S}$ to $\mathbb{B}^{\oplus}$. We do not consider this further here.

Remark 4.3.5. If $\mathcal{S}$ is an inscribed presheaf, then $(T_{\mathcal{S}})_0/\mathcal{S}_0$ is a $\mathbb{B}_0$-module. To define it, it suffices to know just the restriction of $\mathcal{S}$ to the category of finite free square-zero thickenings, and Lemma 4.3.3 shows that knowledge of this restriction is in fact equivalent to knowledge of $(T_{\mathcal{S}})_0/\mathcal{S}_0$.

4.4. **Topologies.** Let τ be a Grothendieck topology on $\mathcal{C}$. Then, since B^{lf} is fibered over $\mathcal{C}$, there is a natural τ-topology also on B^{lf}: a family of morphisms with fixed target is a cover if and only if its image in $\mathcal{C}$ is a cover.

Definition 4.4.1. An *inscribed* τ-prestack/stack/sheaf[6] is an inscribed fibered category that is also a τ-prestack/stack/sheaf on B^{lf}. When the topology is implicit, we will often drop τ from the notation.

The following lemmas are immediate from the definitions. We will use them implicitly with no further comment.

Lemma 4.4.2. *If $\mathcal{S}$ is an inscribed $\tau-$prestack/stack/sheaf, then $\mathcal{S}_0$ is a prestack/stack/sheaf on $\mathcal{C}_\tau$, and if S is a prestack/stack/sheaf on $\mathcal{C}_\tau$, then S^{triv} is an inscribed τ-prestack/stack/sheaf.*

Lemma 4.4.3. *If $\mathcal{S}$ is an inscribed prestack/stack/sheaf, then $T_{\mathcal{S}}$ is an inscribed prestack/stack/sheaf.*

4.5. **Inscribed abelian sheaves and $\mathbb{B}$-modules.** We fix a Grothendieck topology τ on $\mathcal{C}$. For $\mathcal{S}$ an inscribed sheaf, we view an abelian sheaf on $\mathcal{S}$ as a morphism of sheaves on B^{lf} $\mathcal{V} \to \mathcal{S}$ with a zero section $0_{\mathcal{V}} : \mathcal{S} \to \mathcal{V}$ and an addition law $\mathcal{V} \times_{\mathcal{S}} \mathcal{V}$ satisfying the usual compatibilities. We say an abelian sheaf $\mathcal{V}$ on $\mathcal{S}$ is inscribed if $\mathcal{V}$ is inscribed as a presheaf on B^{lf}.

Proposition 4.5.1. *For $\mathcal{S}$ an inscribed sheaf, the category of inscribed abelian sheaves on $\mathcal{S}$ is a full abelian subcategory of the category of abelian sheaves on $\mathcal{S}$.*

Proof. The zero object $\mathcal{S}/\mathcal{S}$ is inscribed, and Lemma 4.1.7 implies that finite products of inscribed abelian sehaves are inscribed and that kernels of maps of inscribed abelian sheaves are inscribed. It remains to see that cokernels are inscribed. Because kernels are inscribed, it suffices to see that quotients are also inscribed.

We will use the following notation: for $\mathcal{F}$ a presheaf on B^{lf} and $\mathcal{B}/B(o) \in B^{\mathrm{lf}}$, we write $\mathcal{F}_{\mathcal{B}}$ for the presheaf on $\mathcal{C}/o$ sending $o' \to o$ to $\mathcal{F}(\mathcal{B} \times_{B(o)} B(o'))$.

Let $\mathcal{V} \subseteq \mathcal{W}$ be inscribed abelian sheaves over $\mathcal{S}$. Suppose given closed immersions $\mathcal{B}_0 \hookrightarrow \mathcal{B}_i$, $i = 1, 2$ lying over $o \in \mathcal{C}$, such that $\mathcal{B} = \mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2$. We then have

$$\begin{aligned}(\mathcal{W}/\mathcal{V})_{\mathcal{B}} &= \mathcal{W}_{\mathcal{B}}/\mathcal{V}_{\mathcal{B}} \\ &= \mathcal{W}_{\mathcal{B}_1} \times_{\mathcal{W}_{\mathcal{B}_0}} \mathcal{W}_{\mathcal{B}_2}/V_{\mathcal{B}_1} \times_{\mathcal{V}_{\mathcal{B}_0}} \mathcal{V}_{\mathcal{B}_2} \\ &= \mathcal{W}_{\mathcal{B}_1}/\mathcal{V}_{\mathcal{B}_1} \times_{\mathcal{W}_{\mathcal{B}_0}/\mathcal{V}_{\mathcal{B}_0}} \mathcal{W}_{\mathcal{B}_2}/\mathcal{V}_{\mathcal{B}_2} \\ &= (\mathcal{W}/\mathcal{V})_{\mathcal{B}_1} \times_{(\mathcal{W}/\mathcal{V})_{\mathcal{B}_0}} (\mathcal{W}/\mathcal{V})_{\mathcal{B}_2}.\end{aligned}$$

[6]A fibered category is a prestack if morphisms between objects are sheaves, and a prestack is a stack if objects also satisfy descent. A presheaf, viewed as a fibered category with discrete fibers, is automatically a prestack, and is a stack if and only if it is a sheaf.

Only the third equality requires some justification: The natural map

$$\left(\mathcal{W}_{\mathcal{B}_1} \times_{\mathcal{W}_{\mathcal{B}_0}} \mathcal{W}_{\mathcal{B}_2}\right) \to \mathcal{W}_{\mathcal{B}_1}/\mathcal{V}_{\mathcal{B}_1} \times_{\mathcal{W}_{\mathcal{B}_0}/\mathcal{V}_{\mathcal{B}_0}} \mathcal{W}_{\mathcal{B}_2}/\mathcal{V}_{\mathcal{B}_2}$$

has kernel $\mathcal{V}_{\mathcal{B}_1} \times_{\mathcal{V}_{\mathcal{B}_0}} \mathcal{V}_{\mathcal{B}_2}$, so it remains to show it is surjective. But, for (a,b) in the image, if we choose on some cover preimages $\tilde{a}$ and $\tilde{b}$, then the images of $\tilde{a}$ and $\tilde{b}$ in $\mathcal{W}_{\mathcal{B}_0}$ differ by an element of $\mathcal{V}_{\mathcal{B}_0}$. Since $\mathcal{V}_{\mathcal{B}_1} \to \mathcal{V}_{\mathcal{B}_0}$ is surjective already as a map of presheaves (it admits a section), we may modify the lift $\tilde{a}$ so that $\tilde{a}$ and $\tilde{b}$ have the same image in $\mathcal{W}_{\mathcal{B}_0}$, so that $(\tilde{a},\tilde{b})$ is a section of $\mathcal{W}_{\mathcal{B}_1} \times_{\mathcal{W}_{\mathcal{B}_0}} \mathcal{W}_{\mathcal{B}_2}$ mapping to (a,b). □

Corollary 4.5.2. *For $\mathcal{S}$ an inscribed sheaf, the category of inscribed sheaves of $\mathbb{B}$-modules on $\mathcal{S}$ is a full abelian subcategory of the category of sheaves of $\mathbb{B}$-modules on $\mathcal{S}$.*

4.6. **Relative tangent bundles and normal bundles.** If $f : \mathcal{Z} \to \mathcal{S}$ is a morphism of inscribed presheaves, we obtain by Proposition 4.3.2 a morphism of $\mathbb{B}$-modules on $\mathcal{Z}$, $df : T_{\mathcal{Z}} \to f^*T_{\mathcal{S}}$.

Definition 4.6.1. If $f : \mathcal{Z} \to \mathcal{S}$ is a morphism of inscribed presheaves, we let $T_{\mathcal{Z}/\mathcal{S}} := \ker df$. If $f : \mathcal{Z} \to \mathcal{S}$ is a morphism of incsribed sheaves, we let $N_{\mathcal{Z}/\mathcal{S}} := \operatorname{coker} df$.

In the setting of Definition 4.6.1, it follows from Corollary 4.5.2 that $T_{\mathcal{Z}/\mathcal{S}}$ and $N_{\mathcal{Z}/\mathcal{S}}$ are inscribed $\mathbb{B}$-modules over $\mathcal{Z}$.

Example 4.6.2. If S is a presheaf on $\mathcal{C}$ and $\mathcal{Z}/S^{\mathrm{triv}}$, then $T_{S^{\mathrm{triv}}} = 0$ so $T_{\mathcal{Z}/S^{\mathrm{triv}}} = T_{\mathcal{Z}}$. In particular, for any inscribed presheaf $\mathcal{S}$, $T_{\mathcal{S}} = T_{\mathcal{S}/B^{\mathrm{lf}}} = T_{\mathcal{S}/\mathcal{S}_0^{\mathrm{triv}}}$, where in the middle B^{lf} is treated as the trivial presheaf on B^{lf}.

4.7. **Inscribed groups.** For $\mathcal{S}$ an inscribed presheaf, an inscribed group over $\mathcal{S}$ is a map of inscribed presheaves $\mathcal{G}/\mathcal{S}$ equipped with an identity section $e : \mathcal{S} \to \mathcal{G}$ and a multiplication law $\mathcal{G} \times_{\mathcal{S}} \mathcal{G} \to \mathcal{G}$ satisfying the usual compatibilities (the multiplication is allowed here to be non-abelian).

Example 4.7.1. Inscribed $\mathbb{B}$-modules are, in particular, inscribed groups.

Lemma 4.7.2. *For $\mathcal{S}$ an inscribed sheaf and $\mathcal{G}/\mathcal{S}$ an inscribed group, $T_{\mathcal{G}/\mathcal{S}}/\mathcal{S}$ admits a canonical inscribed group structure. The structure map $T_{\mathcal{G}/\mathcal{S}} \to \mathcal{G}$ is a surjective homomorphism. It is canonically split by the zero section $\mathcal{G} \to T_{\mathcal{G}/\mathcal{S}}$, and on the kernel $\operatorname{Lie}\mathcal{G} = e^*T_{\mathcal{G}/\mathcal{S}}$, the two natural group structures over $\mathcal{S}$ agree (one as a subgroup of $T_{\mathcal{G}/\mathcal{S}}$, and the other by pull-back of the $\mathbb{B}$-module structure on $T_{\mathcal{G}/\mathcal{S}}$ along $e : \mathcal{S} \to \mathcal{G}$). In particular, $T_{\mathcal{G}/\mathcal{S}} = \mathcal{G} \ltimes \operatorname{Lie}\mathcal{G}$ for the natural $\mathbb{B}$-linear conjugation action on $\operatorname{Lie}\mathcal{G}$.*

Proof. For any $\mathcal{B} \in B^{\mathrm{lf}}$, $\mathcal{G}(\mathcal{B}[\epsilon])/\mathcal{S}(\mathcal{B}[\epsilon])$ is a group over $\mathcal{S}(\mathcal{B}[\epsilon])$, and $T_{\mathcal{G}/\mathcal{S}}(\mathcal{B}[\epsilon])$ is the pullback of this to a group over $\mathcal{S}(\mathcal{B})$ along $0_{T_{\mathcal{S}}} : \mathcal{S}(\mathcal{B}) \to \mathcal{S}(\mathcal{B}[\epsilon])$. This shows $T_{\mathcal{G}/\mathcal{S}}$ admits a canonical inscribed group structure, and it is evident that the structure map to $\mathcal{G}/\mathcal{S}$ is a surjective group homomorphism split by the zero section.

To see the two group structures on $\mathrm{Lie}\,\mathcal{G}$ agree, we first note that the subgroup structure can be written as $dm_e : \mathrm{Lie}\,\mathcal{G} \times_{\mathcal{S}} \mathrm{Lie}\,\mathcal{G} \to \mathrm{Lie}\,\mathcal{G}$, where $m : \mathcal{G} \times_{\mathcal{S}} \mathcal{G} \to \mathcal{G}$ is the multiplication map. This is a $\mathbb{B}$-linear map; in particular we can conlude the two group structures agree because

$$dm_e((a,b)) = dm_e((a,0) + (0,b)) = dm_e((a,0)) + dm_e((0,b)) = a + b.$$

□

Suppose $f : \mathcal{Z} \to \mathcal{S}$ is a map of inscribed presheaves, and $\mathcal{G}/\mathcal{S}$ is an inscribed presheaf. A (right) action of $\mathcal{G}$ on $\mathcal{Z}$ is a map $a : \mathcal{Z} \times_{\mathcal{S}} \mathcal{G} \to \mathcal{Z}$ satisfying the usual axioms. We note that a (right) action of $\mathcal{G}$ on $\mathcal{Z}$ induces a (right) action of $T_{\mathcal{G}/\mathcal{S}}$ on $T_{\mathcal{Z}/\mathcal{S}}$. We write da_e for the induced map $f^* \mathrm{Lie}\,\mathcal{G} \to T_{\mathcal{Z}/\mathcal{S}}$ obtained by pulling back da along $\mathrm{Id}_{\mathcal{Z}} \times e$. Concretely, given a tangent vector $t : \mathcal{B}[\epsilon] \to \mathcal{G}$ restricting to $e : \mathcal{B} \to \mathcal{G}$ and a $z : \mathcal{B} \to \mathcal{G}$, $da_e(t) = \tilde{z} \cdot t$, where $\tilde{z}$ is the constant extension of z to a $\mathcal{B}[\epsilon]$-point of $\mathcal{Z}$.

Proposition 4.7.3. *We fix a topology τ, and let $a : \mathcal{Z} \times_{\mathcal{S}} \mathcal{G} \to \mathcal{Z}$ be a faithful right action of an inscribed group sheaf $\mathcal{G}$ over an inscribed sheaf $\mathcal{S}$ on an inscribed sheaf $\mathcal{Z}$ over $\mathcal{S}$. Then the quotient $\mathcal{Z}/\mathcal{G}$ is an inscribed sheaf over $\mathcal{S}$ and $T_{(\mathcal{Z}/\mathcal{G})/\mathcal{S}} = T_{\mathcal{Z}/\mathcal{S}}/T_{\mathcal{G}/\mathcal{S}}$. In particular, writing $\pi : \mathcal{Z} \to \mathcal{Z}/\mathcal{G}$ for the quotient map, there is a canonical $\mathcal{G}$-equivariant identification*

$$\pi^* T_{(\mathcal{Z}/\mathcal{G})/\mathcal{S}} = \mathrm{coker}(da_e).$$

Proof. Arguing as in the proof of Proposition 4.5.1,

$$\mathcal{Z}_{\mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2}/\mathcal{G}_{\mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2} = \mathcal{Z}_{\mathcal{B}_1} \times_{\mathcal{Z}_{\mathcal{B}_0}} \mathcal{Z}_{\mathcal{B}_2}/\mathcal{G}_{\mathcal{B}_1} \times_{\mathcal{G}_{\mathcal{B}_0}} \mathcal{G}_{\mathcal{B}_2} = \mathcal{Z}_{\mathcal{B}_1}/\mathcal{G}_{\mathcal{B}_1} \times_{\mathcal{Z}_{\mathcal{B}_0}/\mathcal{G}_{\mathcal{B}_0}} \mathcal{Z}_{\mathcal{B}_2}/\mathcal{G}_{\mathcal{B}_2}$$

we conclude that $\mathcal{Z}/\mathcal{G}$ is inscribed. The rest is immediate. □

4.8. **Moduli of sections.**

Definition 4.8.1.

(1) We write $\mathrm{Sm}_{\mathcal{B}}$ for the category whose objects are pairs $(\mathcal{B}, Z/\mathcal{B})$ where $Z/\mathcal{B}$ is a smooth morphisms of schemes/strongly sheaf adic spaces
(2) For $\mathcal{S}$ an inscribed presheaf, a smooth space over $\mathcal{B}$ on $\mathcal{S}$ is a map of fibered categories $Z : \mathcal{S} \to \mathrm{Sm}_{\mathcal{B}}$.
(3) For $\mathcal{S}$ an inscribed presheaf and Z a smooth space over $\mathcal{B}$ on $\mathcal{S}$, we define $\mathcal{B}^* h_Z$ to be the presheaf sending $\mathcal{B} \in B^{\mathrm{lf}}$ to the set of isomorphism classes of pairs (s, f) where $s \in \mathcal{S}(\mathcal{B})$ and $f \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}/\mathcal{B}, \mathcal{Z}(s)/\mathcal{B})$.

Proposition 4.8.2. *For $\mathcal{S}$ an inscribed presheaf and Z a smooth space over $\mathcal{B}$ on $\mathcal{S}$, $\mathcal{B}^* h_Z$ is an inscribed presheaf over $\mathcal{S}$, and there is a canonical identification of $\mathbb{B}$-modules $T_{\mathcal{B}^* h_Z/\mathcal{S}} = \mathcal{B}^* h_{T_{Z/\mathcal{B}}}$ where $T_{Z/\mathcal{B}}$ is the smooth space over $\mathcal{B}$ on $\mathcal{S}$ sending $s \in S(\mathcal{B})$ to the geometric tangent bundle $T_{Z(s)/\mathcal{B}}$ of Example 3.6.3.*

Proof. To see that it is inscribed, fix a $\mathcal{B}_0 \to \mathcal{S}$ and locally free nilpotent thickenings $\mathcal{B}_1, \mathcal{B}_2$ of $\mathcal{B}_0$. Then, for $\mathcal{B} := \mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2$, we want to show

$$h_Z(\mathcal{B}) \to h_Z(\mathcal{B}_1) \times_{h_Z(\mathcal{B}_0)} h_Z(\mathcal{B}_2)$$

is a bijection. Since $\mathcal{S}$ is inscribed it suffices to work over an element of $s \in \mathcal{S}(\mathcal{B})$ corresponding to $(s_1, s_2) \in \mathcal{S}(\mathcal{B}_1) \times \mathcal{S}(\mathcal{B}_2)$ lying over a common s_0 in $\mathcal{S}(\mathcal{B}_0)$, and we need to show

$$h_Z(\mathcal{B})_s = h_Z(\mathcal{B}_1)_{s_1} \times_{h_Z(\mathcal{B}_0)_{s_0}} h_Z(\mathcal{B}_2)_{s_2}.$$

The elements of $h_Z(\mathcal{B})_s$ are given by

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, Z(s)) = \mathrm{Hom}_{\mathrm{Sch}/\mathcal{B}}(\mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2, Z(s)).$$

By the definition of a coproduct, this is equal to

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{B}_1, Z(s)) \times_{\mathrm{Hom}_{\mathcal{B}}(\mathcal{B}_0, Z(s))} \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}_2, Z(s)).$$

But, since $Z(s) \times_{\mathcal{B}} \mathcal{B}_i = Z(s_i)$, this is an element of

$$\mathrm{Hom}_{\mathcal{B}_1}(\mathcal{B}_1, Z(s_1)) \times_{\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_0, Z(s_0))} \mathrm{Hom}_{\mathcal{B}_2}(\mathcal{B}_2, Z(s_2))$$

which is equal, as desired, to

$$h_Z(\mathcal{B}_1)_{s_1} \times_{h_Z(\mathcal{B}_0)_{s_0}} h_Z(\mathcal{B}_2)_{s_2}.$$

Finally, the identity $T_{\mathcal{B}^* h_Z/\mathcal{S}} = \mathcal{B}^* h_{T_{Z/\mathcal{B}}}$ follows from Example 3.6.3. □

Remark 4.8.3. When Spaces is the category of schemes, we can define more generally a scheme over $\mathcal{B}$ on $\mathcal{S}$ and Proposition 4.8.2 holds in this generality; however, we will not need this generality in what follows. When Spaces is the category of strongly sheafy adic spaces, we must restrict to smooth morphisms for two technical reasons: first, we only know we have fiber products for smooth morphisms (we used this fact implicitly in the claim that $\mathrm{Sm}_{\mathcal{B}}$ is a fibered category), and more seriously, in this case we have only defined tangent bundles for smooth morphisms.

4.9. **Restricted categories of thickenings.** As explained in the introduction, some of our main results will only apply after passing to a restricted category of locally free thickenings of Fargues-Fontaine curves where we impose a natural slope condition. We briefly discuss a general framework for this kind of restriction.

Suppose $B^\bullet \subseteq B^{\mathrm{lf}}$ is a full fibered subcategory such that

(1) For any $o \in \mathcal{C}$, $B(o)/B(o) \in B^\bullet$.
(2) For any $\mathcal{B}/B(o) \in B^\bullet$, and any finite free $\mathbb{B}(\mathcal{B})$-module M, the finite locally free square-zero thickening $\mathcal{B}[M]/B(o)$ of $\mathcal{B}/B(o)$ is also an object of $B^\bullet$.

In this context, we can define a $B^\bullet$-inscribed presheaves/fibered categories/sheaves/prestacks/stacks by replacing B^{lf} everywhere above with $B^\bullet$ and only requiring Eq. (4.1.3.1) for those push-outs that are contained in $B^\bullet$. The basic definitions, structures, and results, in the previous sections for B^{lf}-inscribed objects then make sense also for $B^\bullet$-inscribed objects.

We will typically apply this only to $B^\bullet$-inscribed objects that are obtained by restricting a B^{lf}-inscribed object, but for which there is some natural property that only holds only over $\bullet$-thickenings. We note that, in this case, the tangent bundle of the restricted $B^\bullet$ inscribed v-sheaf is simply the restriction of the tangent bundle of the B^{lf} inscribed v-sheaf.

Example 4.9.1. Let $B^\epsilon \subseteq B^{\mathrm{lf}}$ denote the full subcategory whose objects are those finite locally free thickenings isomorphic to $B(o)[M]/B(o)$ for M a finite locally free $\mathbb{B}(B(o)/B(o)) = \mathcal{O}(B(o))$-module. Then, as in Remark 4.3.5, the category of B^ϵ-inscribed presheaves is equivalent, via $\mathcal{S} \mapsto (\mathcal{S}_0, (T_{\mathcal{S}})_0)$, to the category of presheaves on $\mathcal{C}$ equipped with a presheaf of $\mathbb{B}$-modules.

5. Inscribed contexts

In this section we describe the pairs $(\mathcal{C}, B)$ consisting of a category $\mathcal{C}$ and a functor B from $\mathcal{C}$ to schemes or strongly sheafy adic spaces to which we will apply the formalism of Section 4 in the remainder of this work. After recalling some constructions of adic spaces and schemes attached to perfectoid spaces in Section 5.1 and Section 5.2, in Section 5.3 we define these pairs and state their basic properties (see Proposition 5.3.1). In Section 5.4 we revisit the moduli of sections construction of Section 4.8 in these contexts. In particular, we verify that it gives rise to inscribed v-sheaves in the cases of our main interest. We note that the algebraic moduli of sections construction, which applies only to affine schemes, is relatively straightforward, and suffices for most of our main results. The analytic moduli of sections construction gives a common generalization and inscribed upgrade of the diamonds associated to smooth rigid analytic spaces and Fargues-Scholze moduli of sections in a way that incorporates tangent bundles.

5.1. Perfectoid spaces, untilts, and canonical thickenings.

Let Perf be the category of perfectoid spaces in characteristic p and AffPerf $\subseteq$ Perf for the subcategory of affinoid perfectoid spaces. We equip the categories AffPerf $\subseteq$ Perf with the v-topology of [22, Definition 8.1]. We note that, to define a v-stack on Perf, it suffices to give its values on AffPerf.

Recall that $\mathrm{Spd}\mathbb{Q}_p$ is the v-sheaf on Perf sending P to the set of isomophism classes of untilts $P^\sharp/\mathrm{Spa}\mathbb{Q}_p$. Given such an untilt $P^\sharp/\mathrm{Spa}\mathbb{Q}_p$, there is a canonical infinitesimal thickening for each $i \geq 0$, $P^\sharp_{(i)}$. When $P^\sharp = \mathrm{Spa}(A, A^+)$ is affinoid perfectoid,

$$P^\sharp_{(i)} \coloneqq \mathrm{Spa}(A_{(i)}, A^+_{(i)})$$

where the Huber pair $(A_{(i)}, A^+_{(i)})$ is defined as follows. First, we write $\mathbb{B}^+_{\mathrm{dR}}$ and A_{inf} for the usual Fontaine functors, $\theta : \mathbb{B}^+_{\mathrm{dR}}(A) \twoheadrightarrow A$ for the usual Fontaine map, whose kernel is a Cartier divisor, and $\mathrm{Fil}^j\mathbb{B}^+_{\mathrm{dR}}(A) = (\mathrm{Ker}\theta)^j$. Then

$$A_{(i)} \coloneqq \mathbb{B}^+_{\mathrm{dR}}(A)/\mathrm{Fil}^{i+1}\mathbb{B}^+_{\mathrm{dR}}(A) \text{ and } A^+_{(i)} = \theta^{-1}(A^+)$$

and $A_{(i)}$ is equipped with the f-adic topology such that a ring of definition is given by the image of $A_{\mathrm{inf}}(A^{+,\flat})$. Note that $(A_{(i)}, A^+_{(i)})$, by construction, lies over $(\mathbb{Q}_p, \mathbb{Z}_p)$.

Lemma 5.1.1. *The Huber pair* $(A_{(i)}, A^+_{(i)})$ *is strongly sheafy.*

Proof. For any $n \geq 0$, we write $P_{(i),n} = \mathrm{Spa}(A_{(i)}\langle t_1, \ldots, t_n\rangle, A^+_{(i)}\langle t_1, \ldots, t_n\rangle)$. Note that because these are nilpotent thickenings, for any (i), $P_{(i),n}$ has the same underlying topological space and valuations as $P_{(0),n}$ and rational opens are naturally identified.

We must show that the structure presheaf is a sheaf for each $P_{(i),n}$. We argue by induction on i. When $i = 0$, this holds since perfectoid spaces are strongly sheafy. If we fix a generator ξ for $\ker\theta$, then for any $i \geq 1$ we obtain an exact sequence of presheaves

$$0 \to \mathcal{O}_{P_{(i-1),n}} \xrightarrow{\cdot\xi} \mathcal{O}_{P_{(i),n}} \to \mathcal{O}_{P_{(0),n}} \to 0.$$

By the inductive hypothesis, $\mathcal{O}_{P_{(i-1),n}}$ is a sheaf and $\mathcal{O}_{P_{(0),n}}$ is a sheaf. It thus follows that $\mathcal{O}_{P_{(i),n}}$ is a sheaf. □

Outside of the affinoid case, we obtain $P^\sharp_{(i)}$ by glueing.

Remark 5.1.2. The canonical thickenings $P^\sharp_{(i)}$, $i \geq 1$, do not fall under the umbrella of Section 3.4 because one cannot choose a structure morphism $P^\sharp_{(i)} \to P^\sharp$. For example, for $P = \mathrm{Spa}(\mathbb{C}_p)$, the associated augmentation is

$$\mathbb{B}^+_{\mathrm{dR}}(\mathbb{C}_p)/\mathrm{Fil}^2\mathbb{B}^+_{\mathrm{dR}}(\mathbb{C}_p) \xrightarrow{\theta} \mathbb{C}_p$$

which does not admit a *continuous* algebra section (e.g., because $\overline{\mathbb{Q}}_p$ is dense in both the target and the source).

In the affinoid case, for $0 \leq i < \infty$, we write $P^{\sharp-\mathrm{alg}}_{(i)} := \mathrm{Spec}\,\mathcal{O}(P^\sharp_{(i)})$, i.e. $\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}(A)/\mathrm{Fil}^{i+1}\mathbb{B}^+_{\mathrm{dR}}(A)$ when $P^\sharp = \mathrm{Spa}(A, A^+)$. The category of vector bundles on $P^\sharp_{(i)}$ is equal to the category of vector bundles on $P^{\sharp-\mathrm{alg}}_{(i)}$ as both are equivalent to projective modules over $\mathcal{O}(P^\sharp_{(i)})$ (see Section 3.3).

We also write $P^{\sharp-\mathrm{alg}}_{(\infty)} := \mathrm{Spec}\,\lim_i \mathcal{O}(P^\sharp_{(i)})$, i.e. $P^{\sharp-\mathrm{alg}}_{(\infty)} = \mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}(A)$ when $P^\sharp = \mathrm{Spa}(A, A^+)$.

Remark 5.1.3. In general, we could also view the system $(P^\sharp_{(i)})_i$ as a formal adic space, but we will not need this perspective here.

Definition 5.1.4.

(1) For $0 \leq i < \infty$, we write $\square^\sharp_{(i)}$ for the functor from $\mathrm{AffPerf}/\mathrm{Spd}\mathbb{Q}_p$ to strongly sheafy adic spaces

$$P/\mathrm{Spd}\mathbb{Q}_p \mapsto P^\sharp_{(i)}.$$

We also write $\square^\sharp = \square^\sharp_{(0)}$.

(2) For $0 \leq i \leq \infty$, we write $\square^{\sharp-\mathrm{alg}}_{(i)}$ for the functor from AffPerf/Spd$\mathbb{Q}_p$ to schemes
$$P/\mathrm{Spd}\mathbb{Q}_p \mapsto P^{\sharp-\mathrm{alg}}_{(i)}.$$
We also write $\square^{\sharp-\mathrm{alg}} = \square^{\sharp-\mathrm{alg}}_{(0)}$.

(3) We write $\square^{\sharp-\mathrm{alg}}_{(\infty)} \backslash \square^{\sharp-\mathrm{alg}}$ for the functor from AffPerf/Spd$\mathbb{Q}_p$ to schemes
$$P/\mathrm{Spd}\mathbb{Q}_p \mapsto P^{\sharp-\mathrm{alg}}_{(\infty)} \backslash P^{\sharp-\mathrm{alg}}.$$

As noted above, for any $0 \leq i < \infty$, there is a natural equivalence of fibered categories on Perf/Spd$\mathbb{Q}_p$

$$(\square^{\sharp-\mathrm{alg}}_{(i)})^* \mathrm{Vect} = (\square^{\sharp}_{(i)})^* \mathrm{Vect}. \tag{5.1.4.1}$$

Lemma 5.1.5.

(1) *For any any $0 \leq i \leq \infty$, $(\square^{\sharp-\mathrm{alg}}_{(i)})^* \mathrm{Vect}$ is a v-stack.*
(2) *For $0 \leq i < \infty$, $(\square^{\sharp}_{(i)})^* \mathrm{Vect}$ is a v-stack.*
(3) *$(\square^{\sharp-\mathrm{alg}}_{(\infty)} \backslash \square^{\sharp-\mathrm{alg}})^* \mathrm{Vect}$ is a v-prestack.*

Proof. Part (1) is [25, Corollary 17.1.9], and part (2) then follows from **??**.

For (3), note that $P^{\sharp-\mathrm{alg}}_{(\infty)} \backslash P^{\sharp-\mathrm{alg}}$ is affine: indeed, for $P^\sharp = \mathrm{Spa}(A, A^+)$, it is $\mathrm{Spec}\, \mathbb{B}_{\mathrm{dR}}(A)$, where as usual $\mathbb{B}_{\mathrm{dR}}(A)$ is obtained from $\mathbb{B}^+_{\mathrm{dR}}(A)$ by inverting any generator of $\ker\theta$. To that end, we note that $\mathbb{B}_{\mathrm{dR}}$ is a v-sheaf: This holds, e.g., since if we restrict this to Perf/P for any $P/\mathrm{Spd}\mathbb{Q}_p$ with $P^\sharp = \mathrm{Spa}(A, A^+)$ and fix a generator ξ for $\ker\theta$ on $\mathbb{B}^+_{\mathrm{dR}}(A)$, then the restriction to Perf/P is $\bigcup \frac{1}{\xi^i} \mathbb{B}^+_{\mathrm{dR}}$, so it is a v-sheaf since $\mathbb{B}^+_{\mathrm{dR}}$ is (that $\mathbb{B}^+_{\mathrm{dR}}$ is a v-sheaf is part of the case $i = (\infty)$ of (1)). This implies part (3): note that the presheaf of homomorphisms between any $\mathbb{B}_{\mathrm{dR}}$-modules M_1 and M_2 is the presheaf of sections of $M_1^* \otimes M_2$. The latter is a projective module so that its sheaf of sections is a summand of $\mathbb{B}^n_{\mathrm{dR}}$ for some n. □

5.2. **Fargues-Fontaine curves.** For a $E/\mathbb{Q}_p$ a finite extension with residue field $\mathbb{F}_q$ and $P = \mathrm{Spa}(R, R^+) \in \mathrm{AffPerf}/\mathrm{Spd}\mathbb{F}_q$, as in [7, II.1.15] we write
$$Y_{E,P} = \mathrm{Spa}(W_E(R^+), W_E(R^+)) \backslash V([\varpi]p)$$
where ϖ is any pseudouniformizer in R^+. It admits a q-power Frobenius σ, and the Fargues-Fontaine curve is
$$X_{E,P} := Y_{E,P}/\sigma^{\mathbb{Z}}.$$
As in [7, §II.2.3], there is an ample line bundle $\mathcal{O}(1)$ on $X_{E,P}$, and defining $X^{\mathrm{alg}}_{E,P} := \mathrm{Proj} \bigoplus_{i \geq 0} H^0(X_E, \mathcal{O}(i))$, there is a natural map of ringed spaces $X_{E,P} \to X^{\mathrm{alg}}_{E,P}$ such that pullback induces an equivalence

$$\mathrm{Vect}(X^{\mathrm{alg}}_{E,P}) = \mathrm{Vect}(X_{E,P}). \tag{5.2.0.1}$$

that furthermore identifies cohomology groups on both sides.

Note that there is a map $\mathrm{Spd}E \to \mathrm{Spd}\mathbb{F}_q$. For $P/\mathrm{Spd}E$, Fontaine's map θ induces functorial closed immersions over E for $0 \le i < \infty$

$$P^\sharp_{(i)} \hookrightarrow Y_{E,P}, P^\sharp_{(i)} \hookrightarrow X_{E,P}, \text{ and } P^{\sharp-\mathrm{alg}}_{(i)} \hookrightarrow X^{\mathrm{alg}}_{E,P}.$$

It also induces a functorial map

$$P^{\sharp-\mathrm{alg}}_{(i)} \to X^{\mathrm{alg}}_{E,P}$$

which is the algebraization of the formal neighborhod of $P^{\sharp-\mathrm{alg}}$ in $X^{\mathrm{alg}}_{E,P}$.

Definition 5.2.1.

(1) We write $X_{E,\square}$ for the functor from $\mathrm{AffPerf}/\mathrm{Spd}\mathbb{F}_q$ to strongly sheafy adic spaces

$$P/\mathrm{Spd}\mathbb{F}_q \mapsto X_{E,P}.$$

(2) We write $X^{\mathrm{alg}}_{E,\square}$ for the functor from $\mathrm{AffPerf}/\mathrm{Spd}\mathbb{F}_q$ to schemes

$$P/\mathrm{Spd}\mathbb{F}_q \mapsto X^{\mathrm{alg}}_{E,P}.$$

(3) We write $X^{\mathrm{alg}}_{E,\square}\backslash\square^{\sharp-\mathrm{alg}}$ for the functor from $\mathrm{AffPerf}/\mathrm{Spd}E$ to schemes

$$P/\mathrm{Spd}E \mapsto X_{E,P}\backslash P^{\sharp-\mathrm{alg}}.$$

Lemma 5.2.2.

(1) $(X_{E,\square})^*\mathrm{Vect}$ *and* $(X^{\mathrm{alg}}_{E,\square})^*\mathrm{Vect}$ *are both v-stacks. They are equivalent by pullback along the natural transformation of functors to ringed spaces* $X_{E,\square} \to X^{\mathrm{alg}}_{E,\square}$.

(2) $(X_{E,\square}\ \square^{\sharp-\mathrm{alg}})^*\mathrm{Vect}$ *is a v-prestack.*

Proof. For (1), the equivalence follows from the GAGA equivalence Eq. (5.2.0.1), so it suffices to establish the stack property only in the analytic case. To that end, we first note that, for any open $U \subseteq Y_{E,P}$, $U^*\mathrm{Vect}$ is a v-stack on $\mathrm{AffPerf}/P$ by [25, Proof of Proposition 19.5.3]. In particular, since the category of vector bundles on X_E is equivalent to the category of φ-equivariant bundles on Y_E, it follows that $X^*_{E,\square}\mathrm{Vect}$ is a v-stack.

For (2), we note that $X^{\mathrm{alg}}_{E,P}\backslash P^\sharp$ is affine. For $P^\sharp = \mathrm{Spa}(A, A^+)$, its global sections are usually written as $\mathbb{B}_e(A)$. As in the proof of Lemma 5.1.5, it suffices to verify these global section sare a v-sheaf. This follows, e.g., by writing its restriction to any $P/\mathrm{Spd}E$ as the colimit of the global sections presheaves of $\mathcal{O}(n)$ on $X_{E,P}$, which is a v-sheaf by part (1). ☐

5.3. **The pairs $(\mathcal{C}, B)$ that we will use.**

Proposition 5.3.1. *Consider the pairs* $(\mathrm{AffPerf}/S, B)$ *for* (S, B) *as follows:*

(1) $S = \mathrm{Spd}\mathbb{Q}_p$ *and* $B = \square^\sharp_{(i)}$ *or* $\square^{\sharp-\mathrm{alg}}_{(i)}$ *for any* $0 \le i < \infty$

(2) $S = \mathrm{Spd}\mathbb{Q}_p$ *and* $B = \square^{\sharp-\mathrm{alg}}_{(\infty)}$

(3) $S = \mathrm{Spd}\mathbb{Q}_p$ *and* $B = \square^{\sharp-\mathrm{alg}}\backslash\square^{\sharp-\mathrm{alg}}_{(\infty)}$

(4) *For $E/\mathbb{Q}_p$ a finite extension with residue field $\mathbb{F}_q$, $S = \mathrm{Spd}\mathbb{F}_q$ and $B = X_{E,\square}$ or $X_{E,\square}^{\mathrm{alg}}$.*
(5) *For $E/\mathbb{Q}_p$ a finite extension with residue field $\mathbb{F}_q$, $S = \mathrm{Spd}E$ and $B = X_{E,\square}^{\mathrm{alg}} \backslash \square^{\sharp-\mathrm{alg}}$.*

In all cases the presheaf $\mathbb{B}$ on B^{lf} of Definition 4.2.1 is an inscribed v-sheaf. In (1) and (4), there is a canonical equivalence between the categories B^{lf} for the analytic and algebraic versions, identifying the v-sheaf $\mathbb{B}$.

Proof. That $\mathbb{B}$ is inscribed is Proposition 4.2.2 and that it is a v-sheaf follows from Lemma 5.1.5 in cases (1)-(3) and Lemma 5.2.2 in cases (4) and (5). In light of Proposition 3.4.3, the equivalence between the algebraic and analytic categories of thickenings follows in (1) from Eq. (5.1.4.1) and in (4) from Eq. (5.2.0.1) (or the corresponding part of Lemma 5.2.2-(1)). □

5.4. **Moduli of sections.** We consider now a pair $(\mathrm{AffPerf}/S, B)$ as in Proposition 5.3.1. For $\mathcal{S}$ an inscribed sheaf, and Z a smooth scheme or strongly sheaf adic space over $\mathcal{B}$ on $\mathcal{S}$ as in Section 4.8, we write $Z^{\diamond\mathrm{lf}}$ for the presheaf $\mathcal{B}^* h_Z$ over $\mathcal{S}$ of Proposition 5.3.1, i.e.

$$Z^{\diamond\mathrm{lf}}(s \in \mathcal{S}(\mathcal{B})) = \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, Z(s)).$$

Theorem 5.4.1. *Let $(\mathcal{C}, B)$ be one of the algebraic pairs of Proposition 5.3.1, let $\mathcal{S}$ be an inscribed v-sheaf on B^{lf}, let Z be a smooth* affine *scheme over $\mathcal{B}$ on $\mathcal{S}$. Then $Z^{\diamond\mathrm{lf}}$ is an inscribed v-sheaf, and there is a natural identification of inscribed sheaves of $\mathbb{B}$-modules over $Z^{\diamond\mathrm{lf}}$*

$$T_{Z^{\diamond\mathrm{lf}}/\mathcal{S}} = (T_{Z/\mathcal{B}})^{\diamond\mathrm{lf}}.$$

Proof. In Proposition 5.3.1 we showed $Z^{\diamond\mathrm{lf}}$ was an inscribed presheaf, and the made the identification of tangent bundles. Thus it remains only to verify that $Z^{\diamond\mathrm{lf}}$ is a v-sheaf.

It suffices, for each $s : \mathcal{B}_0/B(P/S) \to \mathcal{S}$ and $Z_0 := Z(s)$, to verify that the presheaf on $\mathrm{AffPerf}/P$

$$Q/P \mapsto \mathrm{Hom}_{\mathcal{B}_{0,B(Q/S)}}(\mathcal{B}_{0,B(Q/S)}, Z_{0,B(Q/S)}) = \mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_{0,B(Q/S)}, Z_0)$$

is a v-sheaf. In the cases (1)-(3) and (5) where everything in sight is affine, writing $\mathcal{B}_0 = \mathrm{Spec}\, C$ and $Z_0 = \mathrm{Spec}\, D$, this is

$$Q/P \mapsto \mathrm{Hom}_{\mathbb{B}(P/S)}(D, \mathbb{B}(Q/S) \otimes_{\mathbb{B}(P/S)} C).$$

This is a v-sheaf since $Q/P \mapsto \mathbb{B}(Q/S) \otimes_{\mathbb{B}(P/S)} C$ is a v-sheaf by Lemma 5.1.5 in cases (1)-(3) and Lemma 5.2.2 in case (5) (it is the v-sheaf of sections of $\mathcal{O}_{\mathcal{B}_0}$ viewed as an object of $\mathrm{Vect}(B(P/S))$). In case (4), we may fix an untilt $P^\sharp/\mathrm{Spd}\mathbb{Q}_p$, then deduce the result from that in cases (2)-(4) by writing

$$Z^{\diamond\mathrm{lf}} = (Z|_{X_{E,\square}^{\mathrm{alg}} \backslash \square^{\sharp-\mathrm{alg}}})^{\diamond\mathrm{lf}} \times_{(Z|_{\square^{\sharp-\mathrm{alg}}_{(\infty)} \backslash \square^{\sharp-\mathrm{alg}}})^{\diamond\mathrm{lf}}} (Z|_{\square^{\sharp-\mathrm{alg}}_{(\infty)}})^{\diamond\mathrm{lf}}.$$

□

Theorem 5.4.2. *Let $(\mathcal{C}, B)$ be on of the analytic pairs of Proposition 5.3.1, and let Z be a smooth adic space over $\mathcal{B}$ on $\mathcal{S}$. Then $Z^{\diamond \mathrm{lf}}$ is an inscribed v-stack, and there is a natural identification of $\mathbb{B}$-modules over $Z^{\diamond \mathrm{lf}}$*

$$T_{Z^{\diamond \mathrm{lf}}/\mathcal{S}} = (T_{Z/\mathcal{B}})^{\diamond \mathrm{lf}}.$$

Proof. In Proposition 5.3.1 we showed $Z^{\diamond \mathrm{lf}}$ was an inscribed presheaf, and the made the identification of tangent bundles. Thus it remains only to verify that $Z^{\diamond \mathrm{lf}}$ is a v-sheaf.

It thus remains only to verify that $Z^{\diamond \mathrm{lf}}$ is a v-sheaf. For this it suffices, for each $s : \mathcal{B}_0/B(P/S) \to \mathcal{S}$ and $Z_0 := Z(s)$, to verify that the presheaf on AffPerf$/P$

$$Q/P \mapsto \mathrm{Hom}_{\mathcal{B}_{0,B(Q/S)}}(\mathcal{B}_{0,B(Q/S)}, Z_{0,B(Q/S)}) = \mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_{0,B(Q/S)}, Z_0)$$

is a v-sheaf. We claim this formula defines a v-sheaf on AffPerf$/P$ for any analytic adic space Z_0 over E.

We first treat case (1), so that $B = \square^{\sharp}_{(i)}$, $0 \leq i < \infty$. For a Z_0 affinoid, it follows as in the proof of Theorem 5.4.1 or [22, Lemma 15.1-(ii)], using that $\mathbb{B}$ is in fact a sheaf of topological rings. For any rational open $U \subseteq Z_0$, the functor represented by U is an open sub-functor. Thus, as in [22, §15], we may glue along these open subfunctors to obtain the result for general Z_0.

We now treat case (4), so that $B = X_{E,\square}$. We observe that it suffices to prove the analogous statement over $Y_{E,\square}$, since the property for $X_{E,\square}$ then follows by viewing morphisms from $X_{E,Q} \times_{X_{E,P}} \mathcal{B}_0$ as φ-equivariant morphisms from $Y_{E,Q} \times_{X_{E,P}} \mathcal{B}_0$. We will deduce this statement from the $i = 0$ part of case (1), established above.

To that end, let E_∞ be the completion of the $\mathbb{Z}_p$-subextension of the cyclotomic extension $E(\mu_{p^\infty})$ — E_∞ is a perfectoid field. We write $\Gamma = \mathbb{Z}_p$ for the Galois group, and $\gamma \in \Gamma$ for a topological generator. We first consider the presheaf

$$Q/P \mapsto \mathrm{Hom}(\mathcal{B}_{0,Y_{E,Q}} \times_{\mathrm{Spa}E} \mathrm{Spa}E^{\mathrm{cyc}}, Z_0).$$

It follows from case (1) that this is a v-sheaf — we can apply case (1) here because $Q/P \mapsto Y_{E,Q} \times_{\mathrm{Spa}E} \mathrm{Spa}E_\infty$ is a product preserving functor to perfectoid spaces that sends v-covers to v-covers. We then conclude by observing that the v-sheaf we are interested in is obtained by taking the Γ-invariant sections in this v-sheaf. Indeed, we can check this when Z_0 is affinoid, in which case it reduces to the statement that

$$\mathcal{O}(\mathcal{B}_{0,Y_{E,Q}}) = \mathcal{O}(\mathcal{B}_{0,Y_{E,Q}} \times_{\mathrm{Spa}E} \mathrm{Spa}E^{\mathrm{cyc}})^{\Gamma}.$$

This follows by reducing to the corresponding statement where Y is replaced by the affinoid Y_I for $I \subseteq (0,\infty)$ a compact interval, which follows because

$$\mathcal{O}(\mathcal{B}_{0,Y_{E,I,Q}} \times_{\mathrm{Spa}E} \mathrm{Spa}E_\infty) = \mathcal{O}(\mathcal{B}_{0,Y_{E,I,Q}} \hat{\otimes}_E E_\infty)$$

and, by [27, Proposition 7], there is a direct sum decomposition $E_\infty = E \oplus V$ such that $\gamma - 1$ acts invertibly on V. □

The following example shows that Theorem 5.4.2 encodes both the tangent bundles of smooth rigid analytic varieties over nonarchimedean fields and the Tangent Bundles arising in the Fargues-Scholze Jacobian criterion.

Example 5.4.3.

(1) We work over the pair $(\mathrm{Spd}\mathbb{Q}_p, \square^\sharp)$. Suppose $L/\mathbb{Q}_p$ is a non-archimedean extension and Y/L is a smooth rigid analytic variety. Then,

$$Y \times_{\mathrm{Spa}L} \mathcal{B}$$

is a smooth adic space over $\mathcal{B}$ on $(\mathrm{Spd}L)^{\mathrm{triv}}$, and

$$((Y \times_{\mathrm{Spa}L} \mathcal{B})^{\diamond \mathrm{lf}})_0 = Y^\diamond \text{ and } (T_{(Y\times_{\mathrm{Spa}L}\mathcal{B})^{\mathrm{lf}}})_0 = (T_{Y/\mathrm{Spa}L})^\diamond$$

(2) We work over the pair $(\mathrm{Spd}\mathbb{F}_q, X_{E,\square})$. Suppose $P/\mathrm{Spd}\mathbb{F}_q$ and Z is a smooth adic space over $X_{E,P}$. Then

$$Z \times_{X_{E,P}} \mathcal{B}$$

is a smooth adic space over $\mathcal{B}$ on P^{triv}, and

$$(Z \times_{X_{E,P}} \mathcal{B})^{\mathrm{lf}}_0 = \mathcal{M}_Z \text{ and } (T_{(Z\times_{X_{E,P}}\mathcal{B})^{\mathrm{lf}}})_0 = T_{\mathcal{M}_Z}$$

where $\mathcal{M}_Z$ is the Fargues-Scholze moduli of sections as in [7, IV.4], $\mathcal{M}_Z(Q/P) = \mathrm{Hom}_{X_{E,P}}(X_{E,Q}, Z)$, and $T_{\mathcal{M}_Z}$ is its Tangent Bundle as implicit in [7, IV.4] (cf. [14]), sending $f \in \mathrm{Hom}_{X_{E,P}}(X_{E,Q}, Z)$ to

$$H^0(X_{E,Q}, f^* T_{Z/X_{E,P}}).$$

Remark 5.4.4. The setups of Theorem 5.4.1 and Theorem 5.4.2 also allow for more general constructions: for example, the absolute Banach-Colmez spaces of [7, II.2.2] are sections of smooth adic spaces over $X_{E,\square}$ on $\mathrm{Spd}\overline{\mathbb{F}}_q$, and in the rigid analytic case the formalism allows us to consider ineffective descents of rigid analytic varieties, such as the Breuil-Kisin-Fargues twist $\mathbb{A}^1_{\mathbb{Q}_p}\{1\}$, as smooth adic spaces over $\square^\sharp$ on $\mathrm{Spd}\mathbb{Q}_p$.

Remark 5.4.5. The analytic moduli of sections construction over $\square^\sharp_{(i)}$ for $i \geq 0$ will not be used in the present work, but it plays an important role in our discussion of tangent bundles and Tangent Bundles for p-adic manifold bundles over smooth rigid analytic varieties in [10].

5.5. **Change of context.** We now describe how to move between different inscribed contexts, focusing on the inscribed contexts in Proposition 5.3.1.

We first note that, for a pair $(S/\mathrm{AffPerf}, \mathcal{B})$, if we have a map of v-sheaves on AffPerf, $S' \to S$, then the categories of inscribed v-sheaves over $\mathcal{B}^{\mathrm{lf}}$ equipped with a structure morphism to $(S')^{\mathrm{triv}}$ is naturally equivalent to the category of inscribed v-sheaves over $\mathcal{B}|^{\mathrm{lf}}_{S'}$, compatibly with tangent bundles, etc., and Proposition 5.3.1 still holds if we replace the S in any pair with such an S'. We will use this implicitly below.

Now, for $E/\mathbb{Q}_p$ a finite extension with residue field $\mathbb{F}_q$, we have the natural map $\mathrm{Spd}E \to \mathrm{Spd}\mathbb{F}_q$, and above it, the natural maps

$$\Box^{\sharp}_{(i)} \to X_{E,\Box} \text{ for } 0 \le i < \infty,$$
$$\Box^{\sharp-\mathrm{alg}}_{(i)} \to X^{\mathrm{alg}}_{E,\Box} \text{ for } 0 \le i \le \infty$$
$$X^{\mathrm{alg}}_{E,\Box} \backslash \Box^{\sharp-\mathrm{alg}} \to X^{\mathrm{alg}}_{E,\Box}.$$

Over $\mathrm{Spd}E$ we also have the natural map

$$\Box^{\sharp-\mathrm{alg}}_{(\infty)} \backslash \Box^{\sharp-\mathrm{alg}} \to X^{\sharp-\mathrm{alg}}_{E,\Box} \backslash \Box^{\sharp-\mathrm{alg}}.$$

Thus, for example, we may pullback an inscribed v-sheaf $\mathcal{S}/(\mathrm{Spd}E)^{\mathrm{triv}}$ for the context $(\mathrm{Spd}E, \Box^{\sharp})$ to an inscribed v-sheaf on the inscribed context $(\mathrm{Spd}\mathbb{F}_q, X_E)$ lying over $(\mathrm{Spd}E)^{\mathrm{triv}}$ by

$$\mathcal{S}(\mathcal{X}/X_{E,P}, P \to \mathrm{Spd}E) := \mathcal{S}(\mathcal{X} \times_{X_{E,P}} P^{\sharp}).$$

This construction allows us, e.g., to treat both rigid analytic varieties and Fargues-Scholze moduli of sections as in Example 5.4.3 in a common world. The price one pays is the loss of information about the $\mathbb{B}$-module structure on the tangent bundle — for example, if Z/E is a rigid analytic variety and we pullback $(Z/E)^{\diamond \mathrm{lf}}$ by this construction, then its tangent bundle with respect to the inscribed context $(\mathrm{Spd}\mathbb{F}_q, X_E)$ will only remember the $E^{\diamond \mathrm{lf}}$-module structure rather than the full $\mathcal{O}$-module structure.

5.6. **The slope condition** lf^+**.** In the context of Proposition 5.3.1-(4), we will also consider inscribed v-sheaves, etc., on a restricted category of thickenings $X^{\mathrm{lf}^+}_{E,\Box}$ or equivalently $X^{\mathrm{alg}-\mathrm{lf}^+}_{E,\Box}$ (see Section 4.9). This is the category that was used for the statements of our main results in Section 2 (see Section 2.1); we recall its definition now and observe some basic properties. We will work just with $X^{\mathrm{lf}^+}_{E,\Box}$ — the algebraic version version identical, and is equivalent via the GAGA equivalence of Proposition 5.3.1.

Definition 5.6.1. $X^{\mathrm{lf}^+}_{E,\Box}$ is the full subcategory of $X^{\mathrm{lf}}_{E,\Box}$ whose objects are those $\mathcal{X}/X_{E,P}$ of $X_{E,P}/X_{E,P}$ such that, for $\mathcal{I} : \ker \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_P}$, $\mathcal{I}^n/\mathcal{I}^{n+1}$ is a finite locally free $\mathcal{O}_{X_{E,P}}$-module with non-negative Harder slopes after restriction to $X_{E,\mathrm{Spa}(C,C^+)}$ for any geometric point $\mathrm{Spa}(C,C^+) \to P$.

We now verify that this subcategory satisfies the two conditions enumerated in Section 4.9. First, evidently $X_{E,P}/X_{E,P} \in X^{\mathrm{lf}^+}_{E,\Box}$ for any $P \in \mathrm{Perf}$. Now, suppose given $\mathcal{X}/X_{E,P} \in X^{\mathrm{lf}^+}_{E,\Box}$, and a finite free $\mathcal{O}(\mathcal{X})$-module M. Then, for $\mathcal{I}$ the ideal sheaf of $X_{E,P} \hookrightarrow \mathcal{X}$, the ideal sheaf $\mathcal{I}_M$ of $X_{E,P} \hookrightarrow \mathcal{X}[M]$ is naturally identified with

$$\mathcal{I} \oplus (M \otimes_{\mathcal{O}_{\mathcal{X}}(\mathcal{X})} \mathcal{I}) \cong \mathcal{I} \oplus (\mathcal{I})^{\oplus \mathrm{rank} M}$$

In particular, we find

$$\mathcal{I}_M^n = (\mathcal{I} \oplus (M \otimes_{\mathcal{O}_{\mathcal{X}}(\mathcal{X})} \mathcal{I}))^n = \mathcal{I}^n \oplus (M \otimes_{\mathcal{O}_{\mathcal{X}}(\mathcal{X})} \mathcal{I}^{n-1}),$$

and thus
$$\mathcal{I}_M^n/\mathcal{I}_M^{n+1} = \mathcal{I}^n/\mathcal{I}^{n+1} \oplus (M \otimes_{\mathcal{O}_{\mathcal{X}}(\mathcal{X})} (\mathcal{I}^{n-1}/\mathcal{I}^n)).$$
The slope condition on $\mathcal{I}_{\mathcal{M}}$ thus follows from the slope condition on $\mathcal{I}$.

6. Inscribed vector bundles and G-bundles

We consider one of the following inscribed contexts $(\mathrm{AffPerf}/S, B)$

(1) For L a p-adic field and $0 \leq i \leq \infty$, $(\mathrm{AffPerf}/\mathrm{Spd}L, \square_{(i)}^{\sharp-\mathrm{alg}})$.
(2) For $L/\mathbb{Q}_p$ a finite extension with residue field $\mathbb{F}_q$, $(\mathrm{AffPerf}/\mathrm{Spd}\mathbb{F}_q, X_{E,\square}^{\mathrm{alg}})$.

In the first part of this section we will show that, for G/L a linear algebraic group, the moduli of G-bundles on $\mathcal{B}$ is an inscribed v-stack. The result in the case $i = \infty$ of (1) will be used in the construction and study of the inscribed $\mathbb{B}_{\mathrm{dR}}^+$-affine Grassmannian. The result in the case (2) will give us an inscription on the moduli stack $\mathrm{Bun}G$ of [6]; we emphasize that this is *not* the trivial inscription. The statements are given precisely in Section 6.1. The inscribed property is relatively straightforward and the prestack property can be deduced from the non-inscribed version, but in order to obtain descent we have to redo some of the descent arguments of [25] in our setting.

In the remainder of the section we develop some complements that will be used in the coming sections: In Section 6.3, we briefly discuss the inscribed Banach-Colmez spaces associated to a vector bundle, and in Section 6.4 we discuss the Newton strata on the inscribed classifying stack.

6.1. Vector bundles and the classifying stack.

We write $\mathcal{B}$ for the functor from B^{lf} to schemes over $\mathrm{Spec}\, L$ sending $\mathcal{B}/B(P/S)$ to $\mathcal{B}$, so that $\mathcal{B}^*\mathrm{Vect}$ is the inscribed fibered category over B^{lf} whose objects are pairs $(\mathcal{B}, \mathcal{V})$ where $\mathcal{B} \in B^{\mathrm{lf}}$ and $\mathcal{V}$ is a locally free of finite rank $\mathcal{O}_{\mathcal{B}}$-module.

Theorem 6.1.1. *$\mathcal{B}^*\mathrm{Vect}$ is an inscribed v-stack.*

Proof. That $\mathcal{B}^*\mathrm{Vect}$ is inscribed follows from [8, Theoreme 2.2-(iv)]. We thus must verify it is a v-stack. To see that it is a prestack, observe that if we fix $\mathcal{E}_1$, $\mathcal{E}_2$ in the same fiber, then $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ is the global sections functor of $\mathcal{E}_1^* \otimes \mathcal{E}_2$. By pushing forward to a locally free sheaf of finite rank on B, it follow from Lemma 5.1.5/Lemma 5.2.2 that this is a v-sheaf.

It remains to establish descent. We do not know how to deduce this directly from the descent for $B^*\mathrm{Vect}$, since it is not clear that the descent as a locally free $\mathcal{O}_B$-module with $\mathcal{O}_{\mathcal{B}}$ action is locally free as an $\mathcal{O}_{\mathcal{B}}$-module. However, the proofs of descent in the non-inscribed settings can be adapted to the inscribed setting; we carry this out in the next subsection: in case (1), the result it Lemma 6.2.2, and in case (2), it follows from Lemma 6.2.4 combined with the trivial analytic descent of vector bundles from Y_E to X_E and the GAGA equivalence between vector bundles on X_E and X_E^{alg}. □

Under our assumptions, the functor B and thus also $\mathcal{B}$ factors canonically through schemes over $\mathrm{Spec}\, L$. Thus, for G/L a linear algebraic group, it makes sense to consider also the pull back $\mathcal{B}^*\mathrm{B}G$ of the classifying stack for G. Concretely, $\mathcal{B}^*\mathrm{B}G$ is the fibered category over B^{lf} whose objects are pairs $(\mathcal{B} \in B^{\mathrm{lf}}, \mathcal{G}/B)$ where $\mathcal{G}$ is a G-torsor on $\mathcal{B}$. It will be convenient at various times to use the Tannakian, étale, and geometric perspectives on G-torsors, and we move freely between these.

Theorem 6.1.2. *Suppose G/L is a linear algebraic group. Then $\mathcal{B}^*\mathrm{B}G$ is an inscribed v-stack.*

Proof. That $\mathcal{B}^*\mathrm{B}G$ is a v-stack is immediate from the Tannakian perspective and Theorem 6.1.1. To see that it is inscribed, for $\mathcal{B} = \mathcal{B}_1 \sqcup_{\mathcal{B}_0} \mathcal{B}_2$, we may view each of $G(\mathcal{O}_{\mathcal{B}_\bullet})$, $\bullet = 0, 1, 2$, as an étale sheaf of sections on $\mathcal{B}_{0,\text{ét}}$, so that $\mathrm{B}G(\mathcal{B}_\bullet)$ classifies $G(\mathcal{O}_{\mathcal{B}_\bullet})$-torsors on $\mathcal{B}_{0,\text{ét}}$. But, as in the proof of Theorem 5.4.1, these associated sheaves of sections on $\mathcal{B}_{0,\text{ét}}$ satisfy

$$G(\mathcal{O}_{\mathcal{B}}) = G(\mathcal{O}_{\mathcal{B}_1}) \times_{G(\mathcal{O}_{\mathcal{B}_0})} G(\mathcal{O}_{\mathcal{B}_2}).$$

To give an étale torsor for this group is equivalent to giving étale torsors for $G(\mathcal{O}_{\mathcal{B}_1})$ and $G(\mathcal{O}_{\mathcal{B}_2})$ and an isomorphism of their push-outs to $G(\mathcal{O}_{\mathcal{B}_0})$ — the inverse functor is given by the fiber product of sheaves. □

Remark 6.1.3. Except in the $i = \infty$ case of (1), both of the above results hold also for the analytic version $\mathcal{B}^{\mathrm{an}}$ of $\mathcal{B}$ since there is a GAGA equivalence for the stack of vector bundles $\mathcal{B}^*\mathrm{Vect} \cong (\mathcal{B}^{\mathrm{an}})^*\mathrm{Vect}$.

6.2. **Descent lemmas.** We now prove the descent lemmas that were used in the proof of Theorem 6.1.1. We use the techniques of [25].

Lemma 6.2.1. *Let $P \in \mathrm{Perf}$ be perfectoid with a map $P \to \mathrm{Spd}\mathbb{Q}_p$, and let $\mathcal{B}/P^\sharp$ be a locally free nilpotent thickening of $P^\sharp$. Then, the fibered category on Perf/P sending Q/P to the category of locally free of finite rank $\mathcal{O}_{\mathcal{B}_{Q^\sharp}}$-modules is a v-stack.*

Proof. When $\mathcal{B} = P^\sharp$, this is [25, Lemma 17.1.8]. We will bootstrap from this case.

Suppose Q/P and $\mathcal{E}$ is a locally free of finite rank $\mathcal{O}_{\mathcal{B}_{Q^\sharp}}$. Then, the presheaf on $\mathrm{AffPerf}/Q$,

$$Q'/Q \mapsto H^0(\mathcal{B}_{Q'^\sharp}, \mathcal{E}|_{Q'^\sharp})$$

is a v-sheaf: indeed, it is the v-sheaf of sections of the locally free of finite rank $\mathcal{O}_{Q^\sharp}$-module $\pi_*\mathcal{E}|_{Q^\sharp}$ for $\pi : \mathcal{B}_{Q^\sharp} \to Q^\sharp$, so this follows from [25, Lemma 17.1.8]. In particular, we find that for any two such $\mathcal{E}_1$ and $\mathcal{E}_2$, the presheaf

$$Q'/Q \mapsto \mathrm{Hom}(\mathcal{E}_1|_{\mathcal{B}_{Q'^\sharp}}, \mathcal{E}_2|_{\mathcal{B}_{Q'^\sharp}})$$

is a v-sheaf on Perf/Q since it is the sheaf of sections of $\mathcal{E}_1^* \otimes_{\mathcal{O}_{\mathcal{B}_{Q'^\sharp}}} \mathcal{E}_2$.

It remains to show that all descent data is effective. We may assume $Q = P$. Note that we can use [25, Lemma 17.1.8] to carry out the descent as a locally free $\mathcal{O}_{P^\sharp}$ module with $\mathcal{O}_{\mathcal{B}_{Q^\sharp}}$-action, but it is not clear that the result is locally free as an $\mathcal{O}_{\mathcal{B}_{Q^\sharp}}$-module. However, for P a geometric point, we can deduce the case of a general $\mathcal{B}$ from [23, Lemma 17.1.8] (see below), and in general we will build from the case of a geometric by point adapting the proof of [25, Lemma 17.1.8].

Let $P^\sharp = \mathrm{Spa}(R, R^+)$ and consider a cover $Q \to P$ with $Q^\sharp = \mathrm{Spa}(S, S^+)$. By replacing Q with an open cover, it suffices to descend the trivial module $M = S^n \otimes_R \mathcal{O}(\mathcal{B})$. If we write $\mathcal{O}(\mathcal{B}) = A$, a finite projective R-module, then our descent data is an element $g \in \mathrm{GL}_n((S \hat{\otimes}_R S) \otimes_R A)$ satisfying a cocycle condition (note the second tensor product does not need to be completed since A is a finite projective R-module). We want to show that, locally on $P^\sharp$, we can modify g by a coboundary to obtain the identity matrix.

We first treat the case that $P^\sharp = \mathrm{Spa}(K, K^+)$ is a perfectoid field. In that case, if we write I for the kernel of $A \to K$ and

$$G_i := \ker \mathrm{GL}_n((S \hat{\otimes}_K S) \otimes_K A) \to \mathrm{GL}_n((S \hat{\otimes}_K S) \otimes_K A/I^j)$$

then $G_0/G_1 = \mathrm{GL}_n(S \hat{\otimes}_K S)$ and for $i \geq 1$, $G_i/G_{i+1} = M_n((S \hat{\otimes}_K S) \otimes_K I^i/I^{i+1})$ For G_0/G_1 descent then applies by [25, Lemma 17.1.8] and we can thus modify our descent data g by a cocycle to lie in G_1. Then, the Cech cohomology group involving G_1/G_2 vanishes by [25, Theorem 17.1.3] since it is the Cech cohomology on an affinoid cover of the free $\mathcal{O}$-module $\mathcal{O}^{nm}$, $m = \dim_K A$. Thus we may modify the descent data g by a cocycle to lie in G_2. We repeat this argument until we reach an i large enough that $I^i = 0$ so that $G_i = \{e\}$.

Now, in the general case, fixing $x \in P^\sharp$, we may replace $P^\sharp$ with a rational neighborhood U of x such that I, the kernel of $A \to R$, is free. We fix a basis $i_1, \ldots, i_m$ for I and set $I_0 = R^+ i_1 + \ldots + R^+ i_m \subseteq I$, a free R^+ sub-module of I. Then, for a sufficiently large, $A_0 = R^+ + p^a I_0$ is an open sub-algebra of R (here we use that $(p^a i_1)(p^a i_2) = p^{2a} i_1 i_2$ to see it is a sub-algebra) that is free as an R^+-module. Replacing i_j with $p^a i_j$ and I_0 with $p^a I_0$, we have $A_0 = R^+ + I_0$. Since the descent data is trivial at x, we may choose an element $t(x) \in \mathrm{GL}_n(S \otimes_K A)$ such that $g(x) = (\mathrm{Pr}_1^* t(x))^{-1}(\mathrm{Pr}_2^* t(x))$. Replacing U with a potentially smaller rational neighborhood, we may spread out $t(x)$ to a section t over U. Then, replacing g with $(\mathrm{Pr}_1^* t) g (\mathrm{Pr}_2^* t)^{-1}$ so that $g(x)$ is the identity matrix, we may pass to a potentially smaller rational neighborhood U to assume that g lies in $\mathrm{GL}_n((S^+ \hat{\otimes}_{R^+} S^+) \otimes_{R^+} A_0)$ and even that $g \cong 1 \mod \varpi$ for a pseudo-uniformizer ϖ. Then, $g \bmod \varpi^2$ lies in

$$1 + M_n((S^+ \hat{\otimes}_{R^+} S^+) \otimes_{R^+} (\varpi A_0/\varpi^2 A_0)) \cong (R^+/\omega)^{mn}.$$

By the almost vanishing of the cohomology of the plus-structure sheaf on affinoid perfectoids in [25, Theorem 17.1.3], for any small ϵ we may modify g by a coboundary so that $g \equiv 1 \mod \varpi^{2-\epsilon}$. Iterating, we may modify so that $g \equiv 1 \mod \varpi^{2^k - \epsilon}$ for any k (allowing ϵ to grow but never past 1). The product of the elements we are conjugating by converges, so that we find that g is itself a coboundary and the descent data is trivial over U.

By carrying this out in a neighborhood of any point x we obtain an analytic cover where the descent data is effective, and then we conclude by glueing in the analytic topology. □

As in [25, Corollary 17.1.9], we can also extend over the canonical infinitesimal thickenings. We break this up into two statements; the first giving a geometric statement valid over all finite thickenings $P_{(i)}$, and the second giving a purely algebraic statement valid also over $P^{\mathrm{alg}}_{(\infty)}$.

Lemma 6.2.2. *Let $P/\mathrm{Spd}\mathbb{Q}_p$ be a perfectoid space, and for $i \geq 0$ let $\mathcal{B}/P^{\sharp}_{(i)}$ be a locally free nilpotent thickening. The fibered category over Perf/P sending any Q/P to the category $\mathrm{Vect}(\mathcal{B}_{Q^{\sharp}_{(i)}})$ of locally free of finite rank $\mathcal{O}_{\mathcal{B}_{Q^{\sharp}_{(i)}}}$-modules (for $\mathcal{B}_{Q^{\sharp}_{(i)}} = \mathcal{B} \times_{P^{\sharp}_{(i)}} Q^{\sharp}_{(i)}$) is a v-stack.*

Moreover, if $Q \in \mathrm{AffPerf}/P$, $\mathrm{Vect}(\mathcal{B}_{Q^{\sharp}_{(i)}})$ is equivalent to the category of finite projective $\mathcal{O}(\mathcal{B}_{Q^{\sharp}_{(i)}})$-modules and for any $\mathcal{E} \in \mathrm{Vect}(\mathcal{B}_{Q^{\sharp}_{(i)}})$, $H^j_v(Q, \mathcal{E}) = 0$ for all $j > 0$.

Proof. In general, the equivalence on affinoids with finite projectives is [16, Theorem 8.2.22].

For the remainder of the statement, we argue by induction on i. For $i = 0$, the v-stack property is Lemma 6.2.1, and the vanishing property for higher cohomology on affinoid perfectoids follows from [23, Theorem 17.1.3] since any finite projective module is a direct summand of a finite rank free module.

Now let $i > 0$. We first show that, for any $Q \in \mathrm{Perf}/P$ and $\mathcal{E} \in \mathcal{E} \in \mathrm{Vect}(\mathcal{B}_{Q^{\sharp}_{(i)}})$, $\mathcal{E}$ defines a v-sheaf on Perf/Q whose higher cohomology vanishes on affinoids. To that end, for any $j \leq i$, we write $\mathcal{E}_{(j)} = \mathcal{E} \otimes \mathcal{O}_{\mathcal{B}_{Q^{\sharp}_{(j)}}}$. Then we have an exact sequence

$$0 \to \mathcal{E}_{(i-1)} \to \mathcal{E}_{(i)} \to \mathcal{E}_{(1)} \to 0$$

This induces maps of presheaves of sections on Perf/Q forming a short exact sequence after restriction to $\mathrm{AffPerf}/Q$. Since $\mathcal{E}_{(1)}$ and $\mathcal{E}_{(i-1)}$ both define sheaves on Perf/Q and the higher Cech cohomology of the sheaf defined by $\mathcal{E}_{(i-1)}$ vanishes on any cover in $\mathrm{AffPerf}/Q$, we deduce that $\mathcal{E}_{(i)}$ defines a sheaf on $\mathrm{AffPerf}/Q$, and then by taking the long exact sequence also that its higher cohomology is zero on $\mathrm{AffPerf}/Q$. Since $\mathcal{E}_{(n)}$ already defined a sheaf in the analytic topology of any object in Perf/Q, we conclude it defines a sheaf on Perf/Q.

Now, it follows immediately as in the proof of Lemma 6.2.1 that hom-sets are v-sheaves, so it remains only to establish descent. For $0 \leq j \leq i$, writing G_j for the kernel of

$$\mathrm{GL}_n(\mathcal{O}_{\mathcal{B}_{P^{\sharp}_{(i)}}}) \to \mathrm{GL}_n(\mathcal{O}_{\mathcal{B}_{P^{\sharp}_{(j)}}}),$$

we have $G/G_0 = \mathrm{GL}_n(\mathcal{O}_{\mathcal{B}_{P^{\sharp}}})$ and $G_j/G_{j+1} = 1 + M_n(\mathcal{O}_{\mathcal{B}_{P^{\sharp}_{(0)}}}\{j+1\})$ for $j < i$ and 0 for $j \geq i$. Since each of these subquotients has vanishing H^1 on affinoid perfectoids (the first by the $i = 0$ case of descent), so does $\mathrm{GL}_n(\mathcal{O}_{\mathcal{B}_{P^{\sharp}_{(i)}}})$, and it follows that descent is effective on AffPerf$/P$. Since analytic descent holds already, we obtain the desired descent result. □

Lemma 6.2.3. *Let $P \in$ AffPerf and, for $0 \le i \le \infty$, let $\mathcal{B}$ be a finite locally free nilpotent thickening of* $\mathrm{Spec}\,\mathcal{O}(P_{(i)})$. *Then, the fibered category over* AffPerf$/P$ *sending Q/P to the set of projective $\mathcal{O}(\mathcal{B}) \otimes_{\mathcal{O}(P_{(i)})} \mathcal{O}(Q_{(i)})$-modules is a v-stack. Moreover, for any finite projective $\mathcal{O}(\mathcal{B})$-module M, the associated sheaf of sections*

$$Q/P \mapsto M \otimes_{\mathcal{O}(P_{(i)})} \mathcal{O}_{Q_{(i)}}$$

has vanishing higher cohomology.

Proof. The cases $0 \le i < \infty$ are rephrasings of Lemma 6.2.2. The case ∞ follows by passing to the limit using that

(1) The category of finite projective $\mathcal{O}(\mathcal{B})$-modules is equivalent to the category of compatibles systems of finite projective $\mathcal{O}(\mathcal{B}) \otimes_{\mathcal{O}(\mathcal{P}_{(\infty)})} \mathcal{O}(P_{(i)})$-modules, and
(2) The inverse system of sheaves $(Q/P \mapsto M \otimes_{\mathcal{O}(P_{(\infty)}} \mathcal{O}_{Q_{(i)}})$ whose limit is $Q/P \mapsto M \otimes_{\mathcal{O}(P_{(\infty)})} \mathcal{O}_{Q_{(\infty)}}$ has no higher derived limit.

□

Lemma 6.2.4. *Let $E/\mathbb{Q}_p$ be a finite extension with residue field $\mathbb{F}_q$, let $P \in \mathrm{Perf}/\mathrm{Spd}\mathbb{F}_q$ and let $Y = Y_{E,P}$. Then, for any open $U \subseteq Y$ and any locally free nilpotent thickening $\mathcal{B}/U$, the fibered category over* Perf$/P$ *sending any Q/P to the category of locally free of finite rank $\mathcal{O}_{\mathcal{B}_{Y_{E,Q}}}$-modules (for $\mathcal{B}_{Y_{E,Q}} = \mathcal{B} \times_Y Y_{E,Q}$), is a v-stack.*

Proof. As in the proof of Theorem 5.4.2 and following the proof of [25, Proposition 19.5.3] we can reduce to descent for locally free thickenings of perfectoid spaces. This holds by the $i = 0$ case of Lemma 6.2.2. □

6.3. **Inscribed Banach-Colmez spaces.** In this subsection we work in the inscribed context $(\mathrm{Spd}\mathbb{F}_q, X^{\mathrm{alg}}_{E,\square})$, and write $\mathcal{X}$ for the functor $(X^{\mathrm{alg}}_{E,\square})^{\mathrm{lf}} \to \mathrm{Sch}/E$, $\mathcal{X}/X^{\mathrm{alg}}_{E,P} \mapsto \mathcal{X}$. We write $E^{\diamond\mathrm{lf}}$ for the $(X^{\mathrm{alg}}_{E,\square})^{\mathrm{lf}}$-inscribed v-sheaf $\mathbb{B}$ of Definition 4.2.1,

$$E^{\diamond\mathrm{lf}}(\mathcal{X}) = H^0(\mathcal{X}, \mathcal{O}).$$

Suppose $\mathcal{S}$ is an inscribed v-sheaf and $\mathcal{E} : \mathcal{S} \to \mathcal{X}^*\mathrm{Vect}$. We write $\mathcal{E}_{\mathcal{X}}$ for $\mathcal{E}(\mathcal{X}/X_P \to \mathcal{S})$ below when it will cause no confusion.

We consider the following two functors on $(X^{\mathrm{alg}}_{E,\square})^{\mathrm{lf}}/\mathcal{S}$:

$$\mathrm{BC}(\mathcal{E}) : \mathcal{X}/S \mapsto H^0(\mathcal{X}, \mathcal{E}_{\mathcal{X}}), \text{ and}$$
$$\mathrm{BC}(\mathcal{E}[1]) : \text{the sheafification of } \big(\mathcal{X}/S \mapsto H^1(\mathcal{X}, \mathcal{E}_{\mathcal{X}})\big)$$

which are both naturally $E^{\diamond\mathrm{lf}}$-modules.

Proposition 6.3.1. *Suppose $\mathcal{S}$ is an inscribed v-sheaf, and $\mathcal{E} : \mathcal{S} \to \mathcal{X}^*\mathrm{Vect}$. Then, each of $\mathrm{BC}(\mathcal{E})$ and $\mathrm{BC}(\mathcal{E}[1])$ is an inscribed v-sheaf.*

Proof. For $\mathrm{BC}(\mathcal{E})$, the property of being an inscribed v-sheaf follows from Theorem 5.4.1 applied to the associated geometric vector bundle. Similarly, we obtain the result for $\mathrm{BC}(\mathcal{E}[1])$ by applying Theorem 5.4.2 by pulling back to $\mathcal{X}\times_{X_{E,\square}} Y_{E,\square}$ and realizing $\mathrm{BC}(\mathcal{E}[1])$ as the cokernel of a map of inscribed v-sheaves of abelian groups over Y (as in [7, Proposition II.2.1]). □

Proposition 6.3.2. *Suppose $\mathcal{S}$ is an inscribed v-sheaf and $\mathcal{E}:\mathcal{S}\to\mathcal{X}^*\mathrm{Vect}$ factors through $\mathcal{E}_0:\mathcal{S}\to(\mathcal{X}^*\mathrm{Vect})_0$. If the Harder-Narasimhan slopes of $\mathcal{E}$ at each geometric point $\mathrm{Spd}(C,C^+)\to\mathcal{S}$ are non-negative, then*

$$\mathrm{BC}(\mathcal{E}[1])|_{(X_{E,\square}^{\mathrm{alg}})^{\mathrm{lf}+}}=0.$$

Proof. Given a map $f:\mathcal{X}/X_{E,P}\to\mathcal{S}$, we consider the pullback of $\mathrm{BC}(\mathcal{E})$ to Perf/P. Viewing $f^*\mathcal{E}$ as a vector bundle on $\mathcal{X}$, it follows from our assumption that it is isomorphic to

$$f_0^*\mathcal{E}\otimes_{\mathcal{O}_{X_{E,P}}}\mathcal{O}_{\mathcal{X}}=:\mathcal{V}.$$

The pullback to $\mathrm{BC}(\mathcal{E})$ to Perf/P can thus be viewed as the v-sheafification of

$$(P'\to P)\mapsto H^1(X_{E,P'},\mathcal{V}|_{X_{E,P'}}).$$

By our condition the slopes of $f_0^*\mathcal{E}$ and $\mathcal{O}_{\mathcal{X}}$, the slopes of $\mathcal{V}$ are all nonnegative, thus this v-sheafication is trivial by [6, Proposition II.3.4-(ii)]. □

6.4. **Newton strata.** We maintain the notation of Section 6.3, and fix G/E a connected linear algberaic group. For any $b\in G(\breve{E})$, we have a canonical map $\mathcal{E}_b:\mathrm{Spd}\overline{\mathbb{F}}_q\to\mathcal{X}^*\mathrm{B}G$. It is induced by pullback along $\mathcal{X}\to X_{E,\square}$ from the usual construction

$$\mathcal{E}_b:\mathrm{Spd}\overline{\mathbb{F}}_q\to\mathrm{Bun}G=X_{E,\square}^*\mathrm{B}G$$

which sends $P/\mathrm{Spd}\overline{\mathbb{F}}_q$ to the descent of the trivial G-torsor $\mathcal{E}_{\mathrm{triv}}$ on $Y_{E,P}$ via the isomorphism $\sigma^*\mathcal{E}_{\mathrm{triv}}=\mathcal{E}_{\mathrm{triv}}\xrightarrow{b}\mathcal{E}_{\mathrm{triv}}$. The isomorphism class of $\mathcal{E}_b$ depends only on the σ-conjugacy class $[b]$, and we write $(\mathcal{X}^*\mathrm{B}G)^{[b]}\subseteq\mathcal{X}^*\mathrm{B}G\times\mathrm{Spd}\overline{\mathbb{F}}_q$ for the image of the graph of b (i.e. $\mathcal{E}:\mathcal{X}\to\mathrm{B}G\times\mathrm{Spd}\overline{\mathbb{F}}_q$ factors through $(\mathcal{X}^*\mathrm{B}G)^{[b]}$ if and only if it is v-locally isomorphic to $\mathcal{E}_b$).

Lemma 6.4.1. *$(\mathcal{X}^*\mathrm{B}G)^{[b]}$ is an inscribed v-stack.*

Proof. By Theorem 6.1.2, it remains only to check the inscribed property, and for this the only question is essential surjectivity in Eq. (4.1.3.1). Thus suppose $\mathcal{E}$ is a G-bundle on $\mathcal{X}_1\sqcup_{\mathcal{X}_0}\mathcal{X}_2$ such that $\mathcal{E}|_{\mathcal{X}_i}$ is v-locally isomorphic to b for each i. Then, passing to a sufficiently large cover, we can assume each is isomorphic to $\mathcal{E}_b$. Since the automorphisms of $\mathcal{E}_b|_{\mathcal{X}_1}$ surject onto those of $\mathcal{E}_b|_{\mathcal{X}_0}$, we can then glue to get an isomorphism with $\mathcal{E}_b$ over $\mathcal{X}_1\sqcup_{\mathcal{X}_0}\mathcal{X}_2$. □

Remark 6.4.2. We have $((\mathcal{X}^*\mathrm{B}G)^{[b]})_0=\mathrm{Bun}_G^{[b]}$, for the right hand side as defined in [7, Chapter III] and [13, §4]. This is not "by definition", as the right-hand side is defined by the condition of being isomorphic to $\mathcal{E}_b$ at all geometric points. Nonetheless, for G reductive it follows from the results of [7, Chapter III]. For general G it would follow from the reductive case and the results claimed in [13, §4], however, Lemma 4.1 in the first arXiv version of [13] is not correct as stated[7]; but this equality will still follow from the corrected result to appear shortly.

Recall that $[b]$ is called basic if, for $\mathfrak{g} := \mathrm{Lie}\, G$, equipped with the adjoint action, the associated isocrystal $\mathfrak{g}_b$ is trivial (equivalently, the slope morphism for $[b]$ is central in G).

Proposition 6.4.3. *Suppose* $[b]$ *is basic. Then, the restriction of* $\mathcal{X}^*\mathrm{B}G^{[b]}$ *to* $X_{E,\square}^{\mathrm{lf}^+}$ *is open in the following strong sense:*

1. *The underlying v-stack* $(\mathcal{X}^*\mathrm{B}G)^{[b]} = \mathrm{Bun}_G^{[b]}$ *is an open substack of* Bun_G, *and*
2. $\mathcal{X}^*\mathrm{B}G$ *is the formal neighborhood of* Bun_G^b, *that is,*

$$(\mathcal{X}^*\mathrm{B}G) = (\mathcal{X}^*\mathrm{B}G) \times_{(\mathcal{X}^*\mathrm{B}G)_0 = (\mathrm{Bun}G)} \mathrm{Bun}_G^{[b]}$$

Proof. For G reductive the first claim follows from Remark 6.4.2 and the results of [7], and the general case is deduced from this in [13, Theorem 4.1.2] (note that the proof of Lemma 4.1.1 in [13] is correct in the basic case).

For the second case, we must show that, for $\mathcal{E}$ a G-bundle on $\mathcal{X}/X_{E,P}$, if $\mathcal{E}|_{X_{E,P}}$ is v-locally isomorphic to $\mathcal{E}_b$ then so is $\mathcal{E}$. Writing $\mathcal{X}_{(i)}$ for the thickening corresponding to $\mathcal{O}_{\mathcal{X}}/\mathcal{I}^{i+1}$, we can assume this holds for $\mathcal{E}_{\mathcal{X}_{(i-1)}}$ and then extend to $\mathcal{E}_{\mathcal{X}_{(i)}}$. Passing to a cover, we may assume $\mathcal{E}_{\mathcal{X}_{(i)}} = \mathcal{E}_b|_{\mathcal{X}_{(i)}}$. Then, we claim the isomorphism class of such an extension is classified by an element of $H^1(X_{E,P}, \mathfrak{g} \otimes \mathcal{I}^i/\mathcal{I}^{i+1})$ — this follows because the automorphism group over $\mathcal{X}_{(i+1)}$ of $\mathcal{E}_b|_{\mathcal{X}_{(i)}}$ is an extension of the automorphism group of $\mathcal{E}_b|_{\mathcal{X}_{(i-1)}}$ by $\mathcal{E}(\mathfrak{g}_b) \otimes (\mathcal{I}^i/\mathcal{I}^{i+1})$ which, because of the basic hypothesis, is just $\mathfrak{g} \otimes (\mathcal{I}^i/\mathcal{I}^{i+1})$. The v-local vanishing of this class then follows from [6, Proposition II.3.4-(ii)] (as in Proposition 6.3.2). □

7. The B_{dR}^+ affine Grassmannian

In this section we define the inscribed B_{dR}^+ affine Grassmannian associated to a connected linear algebraic group over a p-adic field and study its basic properties. In Section 7.1 we give the definition and establish its first properties, including the computation of its tangent bundle. In Section 7.2 we define its Schubert cells and compute their tangent bundles. In Section 7.3 we extend the definition of the Bialynicki-Birula map to the inscribed setting, and compute its derivative — this gives Theorem A of the introduction. Finally in Section 7.4 we study the Schubert cells in $G(\mathbb{B}_{\mathrm{dR}})$ and their natural period maps. The main result of that subsection is Theorem 7.4.2, which can be viewed as a toy version of the more refined computations for

[7] For reasons closely related to our need to restrict to lf^+ below

modifications over the Fargues-Fontaine curve of Theorem B/Theorem 9.2.1 and Theorem C/Corollary 9.2.3.

7.0.1. *Notation.* We fix a p-adic field L. We will work in the inscribed context $(\mathrm{Spd}L, \square^{\sharp-\mathrm{alg}}_{(\infty)})$. We write $\mathbb{B}^+_{\mathrm{dR}}$ for the $\square^{\sharp-\mathrm{alg}}_{(\infty)}$-inscribed v-sheaf $\mathbb{B}$ of Definition 4.2.1,

$$\mathbb{B}^+_{\mathrm{dR}} : \mathcal{B}/P^{\sharp-\mathrm{alg}}_{(\infty)} \mapsto \mathcal{O}(\mathcal{B}).$$

It is a sheaf of L-algebras by the natural map $\square^{\sharp-\mathrm{alg}}_{(\infty)} \to \mathrm{Spec}\, L$ on $\mathrm{Spd}L$. We note that the functor $(\mathcal{B}/P^{\sharp-\mathrm{alg}}_{(\infty)}) \mapsto \mathcal{B}$ is naturally identified with $\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}$.

We write $\mathbb{B}_{\mathrm{dR}}$ for the inscribed v-sheaf obtained by change of context Section 5.5 from the inscribed v-sheaf $\mathbb{B}$ with respect to $\square^{\sharp-\mathrm{alg}}_{(\infty)} \backslash \square^{\sharp-\mathrm{alg}}$,

$$\mathbb{B}_{\mathrm{dR}} : \mathcal{B}/P^{\sharp-\mathrm{alg}}_{(\infty)} \mapsto \mathcal{O}\left(\mathcal{B} \times_{P^{\sharp-\mathrm{alg}}_{(\infty)}} (P^{\sharp-\mathrm{alg}}_{(\infty)} \backslash P^{\sharp-\mathrm{alg}})\right).$$

Note that $\mathbb{B}_{\mathrm{dR}}$ is naturally a $\mathbb{B}^+_{\mathrm{dR}}$-algebra.

7.1. **Definition and first properties.** We want to define the B^+_{dR}-affine Grassmannian as a fiber product. Before doing so, we note that the fibered category $(\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}})^*\mathrm{B}G$ is not covered by Theorem 6.1.2. This is because we do not know if descent holds. However, we still have:

Lemma 7.1.1. *Let G/L be a connected linear algebraic group. Then* $(\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}})^*\mathrm{B}G$ *is an inscribed pre-stack.*

Proof. For any $\mathcal{B}/P^{\sharp-\mathrm{alg}}_{(\infty)}$ and G-torsors $\mathcal{G}_1$, $\mathcal{G}_2$ on $\mathcal{B}$, the presheaf of homomorphisms is the moduli of sections for the smooth affine scheme $\mathcal{I}som_{\mathcal{B}}(\mathcal{G}_1, \mathcal{G}_2)$. This is a v-sheaf by Theorem 5.4.1, so $\mathrm{Spec}(\mathbb{B}_{\mathrm{dR}})^*\mathrm{B}G$ is a pre-stack.

It is inscribed by the same argument as in the proof of Theorem 6.1.2. □

Definition 7.1.2. For L a p-adic field and G/L connected linear algebraic group, Gr_G is the prestack on $(\square^{\sharp-\mathrm{alg}}_{(\infty)})^{\mathrm{lf}}$ given by the fiber product

$$\begin{array}{ccc} \mathrm{Gr}_G & \longrightarrow & (\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}})^* BG \\ \downarrow & & \downarrow \\ \mathrm{Spd}L & \xrightarrow{\mathcal{E}_{\mathrm{triv}}} & (\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}})^* BG \end{array}$$

where the right vertical arrow is restriction of G-bundles, $\mathrm{Spd}L$ is equipped with the trivial inscription (i.e. it is the final object), and $\mathcal{E}_{\mathrm{triv}}$ denotes the trivial G-bundle. In other words, $\mathrm{Gr}_G(\mathcal{B})$ classifies G-bundles on $\mathcal{B}$ equipped with a trivialization after restriction to $\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}$.

Note that the automorphism group of any object in Gr_G is trivial, so that passing to isomorphism classes we can and do view it as a presheaf instead of as a fibered category.

Proposition 7.1.3. Gr_G *is an inscribed v-sheaf.*

Proof. It follows from Theorem 6.1.2 and Lemma 7.1.1 that it is a v-sheaf, and it follows from these results combined with Lemma 4.1.7 that it is inscribed. □

There is a natural action of $G(\mathbb{B}_{\mathrm{dR}}) = \mathcal{A}ut(\mathcal{E}_{\mathrm{triv}})$ on Gr_G by changing the trivialization. There is also a canonical point $*_1 : \mathrm{Spd} L \to \mathrm{Gr}_G$ given by $\mathcal{E}_{\mathrm{triv}} \times_{\mathrm{Id}} \mathcal{E}_{\mathrm{triv}}$, i.e. by the trivial bundle with on $\mathrm{Spec}\, \mathbb{B}^+_{\mathrm{dR}}$ with its canonical trivialization after restriction to $\mathrm{Spec}\, \mathbb{B}_{\mathrm{dR}}$.

Proposition 7.1.4. *The action of* $G(\mathbb{B}_{\mathrm{dR}})$ *on* Gr_G *is transitive in the étale topology. In particular, the orbit map for* $*_1$ *induces an identification*

$$\mathrm{Gr}_G = G(\mathbb{B}_{\mathrm{dR}})/G(\mathbb{B}^+_{\mathrm{dR}}),$$

where the quotient can be formed in either the étale or v-topology.

Proof. The first claim follows from the second, so it suffices to show that any G-torsor on $\mathcal{B}/P^{\sharp-\mathrm{alg}}_{(\infty)}$ is trivial after base change to $P'^{\sharp-\mathrm{alg}}_{(\infty)}$ for an étale cover $P' \to P$. The proof is then exactly as in [25, Proposition 19.1.2]. □

We will use this transitivity to compute the tangent bundle of Gr_G. Before making this computation, we introduce some notation.

Definition 7.1.5. For G/L a connected linear algebraic group and $V \in \mathrm{Rep}_G(L)$,

(1) Let V^+_{univ} denote the sheaf of $\mathbb{B}^+_{\mathrm{dR}}$-modules on Gr_G defined as follows: to give a $\mathcal{B}$-point of Gr_G is to give a G-torsor $\mathcal{E}$ on $\mathcal{B}$ with a trivialization after restriction to $\mathcal{B} \times_{P^{\sharp-\mathrm{alg}}_{(\infty)}} P^{\sharp-\mathrm{alg}}_{(\infty)} \backslash P^{\sharp-\mathrm{alg}} = \mathrm{Spec}\, \mathbb{B}_{\mathrm{dR}}(\mathcal{B})$. To such a point, we associate the projective $\mathbb{B}^+_{\mathrm{dR}}(\mathcal{B})$-module
$$\mathcal{E}(V) = H^0(P^{\sharp-\mathrm{alg}}_{(\infty)}, \mathcal{E} \times^G (V \otimes \mathcal{O})).$$

(2) Let $\varphi_{\mathrm{univ}} : V^+_{\mathrm{univ}} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathbb{B}_{\mathrm{dR}} \xrightarrow{\sim} V \otimes_L \mathbb{B}_{\mathrm{dR}}$ send a point as above to the trivialization $\mathcal{E}(V) \otimes_{\mathbb{B}^+_{\mathrm{dR}}(\mathcal{B})} \mathbb{B}_{\mathrm{dR}}(\mathcal{B}) \to V \otimes_L \mathbb{B}_{\mathrm{dR}}(\mathcal{B})$.

Recall that in Section 4.7 we have defined, for any group action a of an inscribed group $\mathcal{G}$ on an inscribed v-sheaf $\mathcal{S}$ its derivative at the identity element $da_e : \mathrm{Lie}\, \mathcal{G} \to T_{\mathcal{S}}$. In the following, we identify $\mathrm{Lie}(G(\mathbb{B}_{\mathrm{dR}})) = \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}$, where $\mathfrak{g} = \mathrm{Lie}\, G(E)$.

Corollary 7.1.6. *For* $a : G(\mathbb{B}_{\mathrm{dR}}) \times_{\mathrm{Spd} L} \mathrm{Gr}_G \to \mathrm{Gr}_G$ *the action map, the derivative* $da_e : \mathfrak{g} \otimes_L \mathbb{B}_{\mathrm{dR}} \to T_{\mathrm{Gr}_G}$ *at the identity section e of* $G(\mathbb{B}_{\mathrm{dR}})$ *is a surjection of inscribed* $\mathbb{B}^+_{\mathrm{dR}}$*-modules over* Gr_G *with kernel* $\varphi_{\mathrm{univ}}(\mathfrak{g}^+_{\mathrm{univ}})$*. It induces a canonical identification of inscribed* $\mathbb{B}^+_{\mathrm{dR}}$*-modules over* Gr_G

$$T_{\mathrm{Gr}_G} = (\mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}})/\mathfrak{g}^+_{\mathrm{univ}} = (\mathfrak{g}^+_{\mathrm{univ}} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathbb{B}_{\mathrm{dR}})/\mathfrak{g}^+_{\mathrm{univ}} = \mathfrak{g}^+_{\mathrm{univ}} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} (\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})$$

where here we use φ_{univ} *to identify the* $\mathbb{B}_{\mathrm{dR}}$*-modules over* Gr_G

$$\mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}} = \mathfrak{g}^+_{\mathrm{univ}} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathbb{B}_{\mathrm{dR}}.$$

Proof. From the transitivity of the action of $G(\mathbb{B}_{\mathrm{dR}})$ on Gr_G established in Proposition 7.1.4, we find da_e is surjective. It remains to compute its kernel.

The stabilizer of a point $(\mathcal{E}, \varphi : \mathcal{E}|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}} \cong \mathcal{E}_{\mathrm{triv}})$ in $G(\mathbb{B}_{\mathrm{dR}})$ is

$$\varphi^{-1} \circ \mathrm{Aut}(\mathcal{E}) \circ \varphi \subseteq \mathrm{Aut}(\mathcal{E}_{\mathrm{triv}}) = G(\mathbb{B}_{\mathrm{dR}}). \tag{7.1.6.1}$$

Note that $\underline{\mathrm{Aut}}(\mathcal{E}_{\mathrm{univ}}) : (\mathcal{E}, \varphi) \mapsto \mathrm{Aut}(\mathcal{E})$ is the moduli of sections of the smooth affine scheme over $\mathcal{B} = \mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}$ on Gr_G, $(\mathcal{E}, \varphi) \mapsto \mathcal{A}ut_{\mathcal{B}}(\mathcal{E})$. It is thus an inscribed v-sheaf over Gr_G by Theorem 5.4.1. Since $\mathcal{A}ut_{\mathcal{B}}(\mathcal{E})$ is naturally identified with $\mathcal{E} \times^G G$ where G on the right is equipped with the adjoint action, it follows from Theorem 5.4.1 that $\mathrm{Lie}\,\underline{\mathrm{Aut}}(\mathcal{E}_{\mathrm{univ}}) = \mathfrak{g}^+_{\mathrm{univ}}$. Using the identification $\mathrm{Lie}(G(\mathbb{B}_{\mathrm{dR}})) = \mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}}$, Eq. (7.1.6.1) identifies the tangent space of the stabilizer with $\varphi_{\mathrm{univ}}(\mathfrak{g}^+_{\mathrm{univ}})$, giving the result. □

7.2. **Schubert cells in the B^+_{dR}-affine Grassmannian.** Let G/L be a connected linear algebraic group, and let $[\mu]$ be a conjugacy class of cocharacters of $G_{\overline{L}}$. For $\mu \in [\mu]$, we write $L(\mu) \subseteq \overline{L}$ for the field of definition of μ. We write $L([\mu]) \subseteq \overline{L}$ for the field of definition of $[\mu]$, i.e. the fixed field of the stabilizer of $[\mu]$ in $\mathrm{Gal}(\overline{L}/L)$. For any $\mu \in [\mu]$, we obtain a point $*_\mu : \mathrm{Spd} L(\mu) \to \mathrm{Gr}_G$ whose value on any $\mathcal{B}$ is $\xi^\mu \cdot *_1$ where ξ is any generator of $\mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}(\mathcal{B})$. This is well defined because, given another generator ξ', $\xi^{-\mu}(\xi')^\mu = (\xi'/\xi)^\mu \in G(\mathbb{B}^+_{\mathrm{dR}})$.

Because the elements of $[\mu]$ are conjugate over $\overline{L}$, we find

Lemma 7.2.1. *The v-sheaf image of $G(\mathbb{B}^+_{\mathrm{dR}}) \cdot *_\mu \subseteq \mathrm{Gr}_G \times \mathrm{Spd} L(\mu)$ in $\mathrm{Gr}_G \times \mathrm{Spd} L([\mu])$ is independent of the choice of $\mu \in [\mu]$.*

Definition 7.2.2. Let $\mathrm{Gr}_{[\mu]} \subseteq \mathrm{Gr}_G \times \mathrm{Spd} L([\mu])$ be the v-sheaf image of $G(\mathbb{B}^+_{\mathrm{dR}}) \cdot *_\mu \subseteq \mathrm{Gr}_G \times \mathrm{Spd} L(\mu)$ in $\mathrm{Gr}_G \times \mathrm{Spd} L([\mu])$ for any choice of $\mu \in [\mu]$.

Proposition 7.2.3. *Let G/L be a connected linear algebraic group. The action of $G(\mathbb{B}^+_{\mathrm{dR}})$ on $\mathrm{Gr}_{[\mu]}$ is transitive in the v-topology and $\mathrm{Gr}_{[\mu]}$ is inscribed. Moreover, the derivative*

$$d\iota_{[\mu]} : T_{\mathrm{Gr}_{[\mu]}} \to \iota^*_{[\mu]} T_{\mathrm{Gr}_G}$$

of the inclusion map

$$\iota_{[\mu]} : \mathrm{Gr}_{[\mu]} \hookrightarrow \mathrm{Gr}_G \times_{\mathrm{Spd} L} \mathrm{Spd} L([\mu])$$

induces, under the identification of Corollary 7.1.6, an isomorphism

$$T_{\mathrm{Gr}_{[\mu]}} = \mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}} / \left(\mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}} \cap \mathfrak{g}^+_{\mathrm{univ}}\right) = \left(\mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}} + \mathfrak{g}^+_{\mathrm{univ}}\right) / \mathfrak{g}^+_{\mathrm{univ}}$$

such that the natural quotient map from $\mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}}$ is the derivative da_e at the identity section of the action map $a : G(\mathbb{B}^+_{\mathrm{dR}}) \times_{\mathrm{Spd} L([\mu])} \mathrm{Gr}_{[\mu]} \to \mathrm{Gr}_{[\mu]}$. Moreover, both

$$\mathfrak{g}^+_{\min} := \mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}} \cap \mathfrak{g}^+_{\mathrm{univ}} \text{ and } \mathfrak{g}^+_{\max} := \mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}} + \mathfrak{g}^+_{\mathrm{univ}}$$

are locally free $\mathbb{B}^+_{\mathrm{dR}}$-modules over $\mathrm{Gr}_{[\mu]}$ whose values on any test object $\mathcal{B}$ can be formed as a literal intersection/sum of $\mathbb{B}^+_{\mathrm{dR}}(\mathcal{B})$-submodules in $\mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}}(\mathcal{B}) = \mathfrak{g}^+_{\mathrm{univ}}(\mathcal{B}) \otimes_{\mathbb{B}^+_{\mathrm{dR}}(\mathcal{B})} \mathbb{B}_{\mathrm{dR}}(\mathcal{B})$.

Proof. For the first part, we can reduce to the case $L = L(\mu)$ for some $\mu \in [\mu]$, so that

$$\mathrm{Gr}_{[\mu]} = G(\mathbb{B}_{\mathrm{dR}}^+) \cdot *_\mu = G(\mathbb{B}_{\mathrm{dR}}^+)/\mathrm{Stab}(*_\mu).$$

and the transitivity is clear. It follows from Lemma 4.1.7 that $\mathrm{Stab}(*_\mu)$ is inscribed, and then from Proposition 4.7.3 that $\mathrm{Gr}_{[\mu]}$ is inscribed. The computation of $d\iota_{[\mu]}$ then follows by comparing with Corollary 7.1.6.

Finally, we verify the claims about $\mathfrak{g}^+_{\mathrm{min}}$ and $\mathfrak{g}^+_{\mathrm{max}}$. For $\mathfrak{g}^+_{\mathrm{min}}$, its formation on any test object is clearly as a literal intersection, so it suffices to show it is locally free. But this follows from the v-stack property of Theorem 6.1.2, since it is easily seen to be locally free after passage to a v-cover where $\mathfrak{g}^+_{\mathrm{univ}} = g \cdot *_{[\mu]}$ for $g \in G(\mathbb{B}_{\mathrm{dR}}^+)$. For $\mathfrak{g}^+_{\mathrm{max}}$, the same argument shows it is locally free, but we must explain why its formation on any test object is a literal sum (since a priori there is a v-sheafification). This follows from the exact sequence

$$0 \to \mathfrak{g}^+_{\mathrm{min}} \to \mathfrak{g}^+_{\mathrm{univ}}(\mathcal{B}) \oplus \mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}}^+ \to \mathfrak{g}^+_{\mathrm{max}} \to 0$$

and the v-acyclicity of locally free $\mathbb{B}_{\mathrm{dR}}^+$-modules given by the second part of Lemma 6.2.2 (applied to $\mathfrak{g}^+_{\mathrm{min}}$ in the long exact sequence of v-cohomology). □

Remark 7.2.4. Note that, in [25, 13], the Schubert cells in the $\mathbb{B}_{\mathrm{dR}}^+$-affine Grassmannian on $\mathrm{AffPerf}/\mathrm{Spd}L$ are defined to consist of those sections whose restriction to any geometric point lies in the orbit of $*_\mu$. With this definition, it is only clear that $(\mathrm{Gr}_{[\mu]})_0$ is contained in the Schubert cell of loc. cit., not that they are equal. However, in the reductive case it follows from the final displayed equation in [7, Proposition VI.2.4] that they are equal, and moreover that the action of $G(\mathbb{B}_{\mathrm{dR}}^+)_0$ on $(\mathrm{Gr}_{[\mu]})_0$ is transitive in the étale topology. Bootstrapping off of this, one can also treat the non-reductive case; this argument should appear in a revised version of [13].

Note that this is still not enough to show that in the inscribed setting we could make the definition of the Schubert cell by only considering thickenings over geometric points (although, by a restriction of scalars argument, once the case of an arbitrary connected linear algebraic group is handled then it is enough if we restrict from locally free nilpotent thickenings to only considering L-constant nilpotent thickenings, i.e. those base changed from L). It is evident from the arguments above that the key property for computing the tangent bundles is that the action be transitive, which is why we have adopted the definition above rather than the pointwise definition.

Remark 7.2.5. If G is reductive, then as a corollary of Proposition 7.2.3, we find that $T_{\mathrm{Gr}_{[\mu]}}$ is v-locally isomorphic to

$$\bigoplus_{\alpha} \mathbb{B}_{\mathrm{dR}}^+/\mathrm{Fil}^{\langle \alpha, [\mu] \rangle}\mathbb{B}_{\mathrm{dR}}^+$$

where the sum is over any choice of positive roots α for $G_{\overline{E}}$.

7.3. **The Bialynicki-Birula map.** In the following, we write $\mathcal{O}$ for the inscribed v-sheaf over $\mathrm{Spd}L$, $\mathcal{B}/P^{\sharp\,\mathrm{alg}}_{(\infty)} \mapsto \mathcal{O}(\mathcal{B}\times_{P^{\sharp-\mathrm{alg}}_{(\infty)}} P^{\sharp-\mathrm{alg}})$. It is the inscribed v-sheaf $\mathbb{B}$ associated to the pair $(P^{\sharp-\mathrm{alg}}, \mathrm{Spd}L)$ viewed in our setting by change of context as in Section 5.5 along $P^{\sharp-\mathrm{alg}} \to P^{\sharp-\mathrm{alg}}_{(\infty)}$.

For G/L a connected linear algebraic group and $V \in \mathrm{Rep}_G(L)$ we have the universal $\mathbb{B}^+_{\mathrm{dR}}$-lattice $V^+_{\mathrm{univ}} \subseteq V \otimes_L \mathbb{B}_{\mathrm{dR}}$ over Gr_G. This lattice induces a natural filtration of $V \otimes_L \mathcal{O}$ by $\mathcal{O}$-modules

$$\begin{aligned}\mathrm{Fil}^i_{V^+_{\mathrm{univ}}}(V\otimes_E \mathcal{O}) &:= \left(\mathrm{Fil}^i\mathbb{B}^+_{\mathrm{dR}}\cdot V^+_{\mathrm{univ}} \cap V\otimes_E \mathbb{B}^+_{\mathrm{dR}}\right)/\left(\mathrm{Fil}^i\mathbb{B}^+_{\mathrm{dR}}\cdot V^+_{\mathrm{univ}} \cap V\otimes_E \mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}\right)\\ &\subseteq (V\otimes_E \mathbb{B}^+_{\mathrm{dR}})/\left(V\otimes_E \mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}\right) = V\otimes_E \mathcal{O}.\end{aligned}$$

This filtration may not be by locally free modules and even when it is it may not be an exact functor from $\mathrm{Rep}_G(L)$ to filtered $\mathcal{O}$-modules. After restricting to a Schubert cell, however, it is. This can be verified after passing to v-cover, which, by definition of the Schubert cell, can be chosen so that that the filtration is isomorphic to that defined by $\star_\mu$ for some $\mu \in [\mu]$. Computing directly one finds that this latter is the filtration $\mathrm{Fil}_{\mu^{-1}}$ over $L(\mu)$, where for any cocharacter τ we define the associated filtration by

$$\mathrm{Fil}^i_\tau(V) = \bigoplus_{j\geq i} V[j], \text{ for } V[j] \text{ the isotypic subspace where } \tau(z) \text{ acts as } z^j.$$

Recall that for any conjugacy class of cocharacters $[\tau]$ there is a flag variety $\mathrm{Fl}_{[\tau]}/L([\tau])$ parameterizing filtrations on the trivial G-torsor that are of type $[\tau]$, i.e. locally isomorphic to Fil_τ for $\tau \in [\tau]$.

Theorem 7.3.1. *For any conjugacy class of cocharacters $[\mu]$ of $G_{\overline{L}}$, the restriction of $V \mapsto \mathrm{Fil}^\bullet_{V^+_{\mathrm{univ}}}(V\otimes\mathcal{O})$ to $\mathrm{Gr}_{[\mu]}$ is a filtration of the trivial G-torsor of type $[\mu^{-1}]$. The resulting map $\mathrm{BB}: \mathrm{Gr}_{[\mu]} \to \mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu^{-1}]}$ is equivariant along the natural map $G(\mathbb{B}^+_{\mathrm{dR}}) \twoheadrightarrow G(\mathcal{O})$, and its derivative fits into the commuting diagram*

$$\begin{array}{ccc} T_{\mathrm{Gr}_{[\mu]}} & = & \mathfrak{g}\otimes_L \mathbb{B}^+_{\mathrm{dR}}/\left(\mathfrak{g}\otimes_L \mathbb{B}^+_{\mathrm{dR}} \cap \mathfrak{g}^+_{\mathrm{univ}}\right) \\ \Big\downarrow dBB & & \Big\downarrow \\ & & \mathfrak{g}\otimes_L \mathbb{B}^+_{\mathrm{dR}}/\left(\mathfrak{g}\otimes_L \mathbb{B}^+_{\mathrm{dR}} \cap \mathfrak{g}^+_{\mathrm{univ}} + \mathfrak{g}\otimes_L \mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}\right) \\ & & \| \\ \mathrm{BB}^* T_{\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu^{-1}]}} & = & (\mathfrak{g}\otimes_L \mathcal{O})/\mathrm{Fil}^0(\mathfrak{g}\otimes_L\mathcal{O}) \end{array}$$

where the top horizontal equality is from Proposition 7.2.3 and the bottom horizontal equality follows from Theorem 5.4.2 and the usual computation of the tangent bundle of the partial flag variety $\mathrm{Fl}_{[\mu^{-1}]}$.

Proof. We have explained the existence of BB above. The equivariance follows from the construction, and then the computation of the derivative is an immediate consequence since the tangent bundles of $\mathrm{Gr}_{[\mu]}$ and $T_{\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu^{-1}]}}$ are both expressed in the form given by differentiating the group actions. □

The computation of $d\mathrm{BB}$ is closely related to Griffiths transversality.

Corollary 7.3.2. *Suppose G/L is a connected linear algebraic group, M/L is a non-archimedean extension, and S/M is a smooth rigid analytic variety. If $f:(S/M)^{\diamond\mathrm{lf}}\to \mathrm{Gr}_{[\mu]}\times_{\mathrm{Spd}L}\mathrm{Spd}M$ is a map of inscribed v-sheaves over $\mathrm{Spd}M$, then $\mathrm{BB}\circ f$ satisfies Griffiths transversality for the trivial connection on the trivial G-torsor, i.e. $d(\mathrm{BB}\circ f)$ factors through*

$$\mathrm{gr}^{-1}(\mathfrak{g}\otimes_L\mathcal{O})=\mathrm{Fil}^{-1}(\mathfrak{g}\otimes_L\mathcal{O})/\mathrm{Fil}^0(\mathfrak{g}\otimes_L\mathcal{O})\subseteq(\mathrm{BB}\circ f)^*T_{\mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu]}}$$

Proof. Suppose given $f:(S/M)^{\diamond\mathrm{lf}}\to\mathrm{Gr}_{[\mu]}\times_{\mathrm{Spd}L}\mathrm{Spd}M$. Then df is a map of $\mathbb{B}^+_{\mathrm{dR}}$-modules over $(S/M)^{\diamond\mathrm{lf}}$, and because $T_{(S/M)^{\diamond\mathrm{lf}}}$ is annihilated by $\mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}$ we find that df factors through the part of $f^*T_{\mathrm{Gr}_{[\mu]}}$ annihilated by $\mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}$. Using the description of Proposition 7.2.3, this is given by

$$\left(\mathfrak{g}\otimes\mathbb{B}^+_{\mathrm{dR}}\cap\mathrm{Fil}^{-1}\mathbb{B}^+_{\mathrm{dR}}\cdot\mathfrak{g}^+_{\mathrm{univ}}\right)/\left(\mathfrak{g}\otimes\mathbb{B}^+_{\mathrm{dR}}\cap\mathfrak{g}^+_{\mathrm{univ}}\right).$$

It then follow from Theorem 7.3.1 that $d(\mathrm{BB}\circ f)$ factors through

$$\mathrm{Fil}^{-1}(\mathfrak{g}\otimes\mathcal{O})/\mathrm{Fil}^0(\mathfrak{g}\otimes\mathcal{O}),$$

i.e. that $f\circ\mathrm{BB}$ satisfies the Griffiths transversality condition for the trivial connection on the trivial G-torsor. □

Remark 7.3.3. When M is also a p-adic field, in which case we may as well assume $M=L$, it seems reasonable to expect that conversely any map to $(S/L)^{\diamond\mathrm{lf}}\to(\mathrm{Fl}_{[\mu^{-1}]}/L)^{\diamond\mathrm{lf}}$ satisfying Griffiths transversality for the trivial connection on the trivial G-torsor factors uniquely through $\mathrm{Gr}_{[\mu]}$. On constant nilpotent thickenings, this can be deduced from [13, Theorem 5.0.3-(4)] applied to various restrictions of scalars groups that appear.

Morally, the reason one might expect such an equivalence when $M=L$ is as follows: first, BB induces a bijection $\mathrm{Gr}_{[\mu]}(\overline{L})=\mathrm{Fl}_{[\mu^{-1}]}(\overline{L})$ (see, e.g., [12]). Second, by a consideration of Hodge-Tate weights, $(d\mathrm{BB})_0$ becomes an isomorphism after push-forward to the étale site of S. One can then imagine that to lift f to Gr_G we must first lift a base point and then lift its derivative and integrate from the lifted base point, but there is now a unique lift of any classical base point and a unique lift of the derivative. Both of these uniqueness statements fail over more general fields, and indeed when the cocharacter is non-minuscule and L is perfectoid one finds positive dimension rigid analytic varieties contained entirely in a single fiber of BB.

7.4. Schubert cells in $G(\mathbb{B}_{\mathrm{dR}})$.

Note that we have two natural right actions of $G(\mathbb{B}_{\mathrm{dR}})$ on itself: the action a_1 by right multiplication and the action a_2 by left multiplication by the inverse. We also have two natural maps $\pi_1,\pi_2:G(\mathbb{B}_{\mathrm{dR}})\to\mathrm{Gr}_G$ defined

$$\pi_1(c)=c\cdot *_1\text{ and }\pi_2(c)=c^{-1}\cdot *_1$$

Definition 7.4.1. Let G/L be a connected linear algebraic group and $[\mu]$ a conjugacy class of cocharacters of $G_{\overline{L}}$.

$$C_{[\mu]} := \pi_1 \times_{\mathrm{Gr}_G} \mathrm{Gr}_{[\mu]} = \pi_2 \times_{\mathrm{Gr}_G} \mathrm{Gr}_{[\mu^{-1}]}. \tag{7.4.1.1}$$

By Lemma 4.1.7, $C_{[\mu]}$ is an inscribed v-sheaf. Note that, if we fix $\mathcal{B}/\mathrm{Spd}L$, a lift to $\mathcal{B} \to \mathrm{Spd}L(\mu)$, and a generator ξ of $\mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}(\mathcal{B})$, then $C_{[\mu]}|_{\mathcal{B}}$ is the two-sided orbit $G(\mathbb{B}^+_{\mathrm{dR}}) \cdot \xi^\mu \cdot G(\mathbb{B}^+_{\mathrm{dR}})$. One could adapt this into another definition of $C_{[\mu]}$ similar to the definition of $\mathrm{Gr}_{[\mu]}$ used above.

Theorem 7.4.2. *The actions a_1 and a_2 restrict to actions of $G(\mathbb{B}^+_{\mathrm{dR}})$ on $C_{[\mu]}$ such that:*

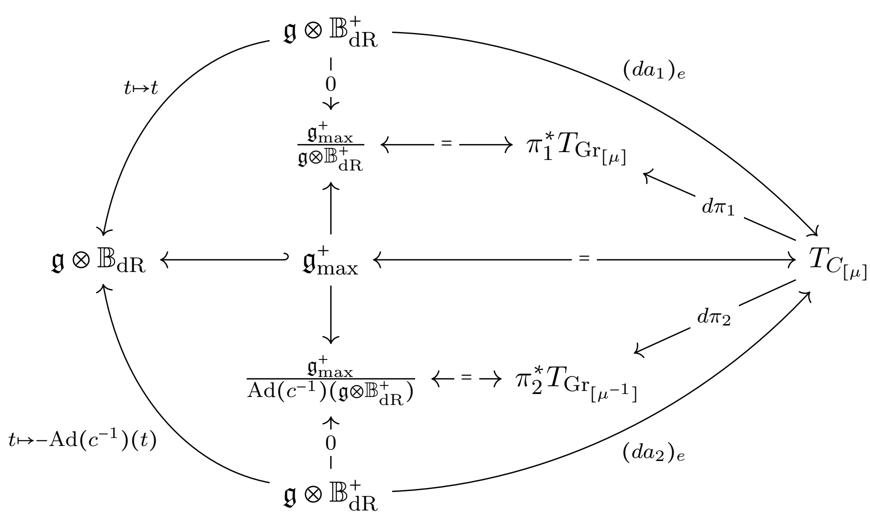
1. *the map π_1 is a $G(\mathbb{B}^+_{\mathrm{dR}})$-torsor over $\mathrm{Gr}_{[\mu]}$ for the action a_1 and is equivariant for the action a_2, and*
2. *the map π_2 is a $G(\mathbb{B}^+_{\mathrm{dR}})$-torsor over $\mathrm{Gr}_{[\mu^{-1}]}$ for the action a_2 and is equivariant for the action a_1.*

The $\mathbb{B}^+_{\mathrm{dR}}$-module on $C_{[\mu]}$

$$\mathfrak{g}^+_{\max} := \mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}} + \mathrm{Ad}(c^{-1})(\mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}}) \overset{\mathrm{Ad}(c)}{\cong} \mathrm{Ad}(c)(\mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}}) + \mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}}.$$

is naturally identified with the preimage under π_1 or π_2 of $\mathfrak{g}^+_{\max}$ of Proposition 7.2.3. In particular, it is locally free of finite rank and can be formed as literal sum of modules on any test object as in Proposition 7.2.3.

The product action a of $G(\mathbb{B}^+_{\mathrm{dR}}) \times G(\mathbb{B}^+_{\mathrm{dR}})$ is transitive, and da_e induces an isomorphism of $\mathbb{B}^+_{\mathrm{dR}}$-modules over $C_{[\mu]}$ fitting into the commutative diagram



Proof. The equivariant torsor structures for π_1 and π_2 are immediate from the definitions and Proposition 7.2.3, and so is the transitivity of the product action a.

The description of $\mathfrak{g}^+_{\max}$ in terms of π_1 and π_2 follows from the definition of π_1 and π_2 and Proposition 7.2.3. Indeed, π_1 classifies the trivial G-torsor on $\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}$ equipped with the trivialization on $\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}$ given by left multiplication by c, while π_2 classifies the trivial G-torsor on $\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}$ equipped with the the trivialization on $\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}$ given by left multiplication by c^{-1}.

The transitivity of the product action a induces a surjection

$$da_e : \mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}} \oplus \mathfrak{g} \otimes \mathbb{B}^+_{\mathrm{dR}} \twoheadrightarrow T_{C_{[\mu]}}.$$

If we compose with $T_{C_{[\mu]}} \hookrightarrow T_{G(\mathbb{B}_{\mathrm{dR}})}|_{C_{[\mu]}}$ and identify the latter with $\mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}}$ using right-invariant vector fields, then da_e is given by

$$(t_1, t_2) \mapsto t_1 - \mathrm{Ad}(c^{-1})(t_2).$$

Indeed, we have $t_2^{-1} c t_1 = c(c^{-1} t_2^{-1} c) t_1$. The image of da_e is thus $\mathfrak{g}^+_{\max}$, giving the claimed isomorphism, and the commutativity of the diagram follows also from this computation and comparison with Proposition 7.2.3. □

8. Modifications

In this section we discuss modifications of G-torsors in the inscribed setting. In particular, in Section 8.1 we explain the fundamental exact sequences of p-adic Hodge theory in the inscribed setting, which play a key role in the computation of tangent bundles for the moduli of modifications in the sections to come. After some preliminary discussion of automorphism groups of G-bundles in Section 8.2, we then construct the inscribed Hecke correpsondence in Section 8.3. Using the inscribed Hecke correspondence, we construct the inscribed generalized Newton strata of the B^+_{dR}-affine Grassmannian and its Schubert cells in Section 8.4.

8.0.1. *Notation.* Let $E/\mathbb{Q}_p$ be a finite extension. In this section, we work in the inscribed context $(\mathrm{Spd}\mathbb{F}_q, X_{E,\square})$.

We will need to consider the sheaf $\mathbb{B}$ not just for $(X^{\mathrm{alg}}_{E,\square})^{\mathrm{lf}}$, but also for from related inscribed contexts by change of base. To disambiguate, we begin by fixing some names for these sheaves.

- We write $E^{\diamond_{\mathrm{lf}}}$ for the $(X^{\mathrm{alg}}_{E,\square})^{\mathrm{lf}}$-inscribed v-sheaf $\mathbb{B}$ of Definition 4.2.1,
$$E^{\diamond_{\mathrm{lf}}}(\mathcal{X}) = H^0(\mathcal{X}, \mathcal{O}).$$
- We write $\mathbb{B}_e$ for the $(X^{\mathrm{alg}}_{E,\square})^{\mathrm{lf}}$-inscribed v-sheaf $\mathbb{B}$ over $\mathrm{Spd}E$ associated, by change of base as in Section 5.5, to $\mathbb{B}$ on $(X_{E,P} \backslash P^\sharp)^{\mathrm{lf}}$:
$$\mathbb{B}_e(\mathcal{X}/X^{\mathrm{alg}}_{E,P}, P/\mathrm{Spd}E) = H^0(\mathcal{X}_{X^{\mathrm{alg}}_{E,P} \backslash P^{\sharp-\mathrm{alg}}}, \mathcal{O}).$$
- We write $\mathbb{B}^+_{\mathrm{dR}}$ for the $(X^{\mathrm{alg}}_{E,\square})^{\mathrm{lf}}$-inscribed v-sheaf over $\mathrm{Spd}E$ associated, by change of base as in Section 5.5, to $\mathbb{B}$ on $(\square^{\sharp-\mathrm{alg}}_{(\infty)})^{\mathrm{lf}}$:
$$\mathbb{B}^+_{\mathrm{dR}}(\mathcal{X}/X^{\mathrm{alg}}_{E,P}, P/\mathrm{Spd}E) = H^0(\mathcal{X}_{P^{\sharp-\mathrm{alg}}_{(\infty)}}, \mathcal{O})$$

- We write $\mathbb{B}_{\mathrm{dR}}$ for the $(X^{\mathrm{alg}}_{E,\square})^{\mathrm{lf}}$-inscribed v-sheaf over $\mathrm{Spd}E$ associated, as in Section 5.5, to $\mathbb{B}$ on $(\square^{\sharp-\mathrm{alg}}_{(\infty)}\backslash\square^{\sharp-\mathrm{alg}})^{\mathrm{lf}}$,

$$\mathbb{B}^{+}_{\mathrm{dR}}(\mathcal{X}/X^{\mathrm{alg}}_{E,P}, P/\mathrm{Spd}E) = H^0(\mathcal{X}_{P^{\sharp-\mathrm{alg}}_{(\infty)}\backslash P^{\sharp-\mathrm{alg}}_{(\infty)}}, \mathcal{O}).$$

Note that $\mathrm{Spec}\,\mathbb{B}_e$ (resp. $\mathrm{Spec}\,\mathbb{B}^{+}_{\mathrm{dR}}$, resp. $\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}$) is canonically identified with the functor sending $(\mathcal{X}/X_{E,P}, P/\mathrm{Spd}E)$ to $\mathcal{X}_{X_{E,P}}\backslash\mathcal{X}_{P^{\sharp-\mathrm{alg}}}$ (resp. $\mathcal{X}_{P^{\sharp-\mathrm{alg}}_{(\infty)}}$, resp. $\mathcal{X}_{P^{\sharp-\mathrm{alg}}_{(\infty)}}\backslash\mathcal{X}_{P^{\sharp-\mathrm{alg}}}$). We will sometimes write $\mathcal{X}\backslash\infty$ for $\mathrm{Spec}\,\mathbb{B}_e$.

8.1. **Fundamental exact sequences.** If $\mathcal{S}$ is an inscribed v-sheaf and $\mathcal{E}: \mathcal{S}\to\mathcal{X}^*\mathrm{Vect}$, recall that in Section 6.3 we have defined $\mathrm{BC}(\mathcal{E})$ and $\mathrm{BC}(\mathcal{E}[1])$. If $\mathcal{S}/\mathrm{Spd}E$, then we also consider, for $\mathbb{B}=\mathbb{B}_e, \mathbb{B}^{+}_{\mathrm{dR}}$, or $\mathbb{B}_{\mathrm{dR}}$,

$$\mathcal{E}\boxtimes\mathbb{B}: \mathcal{X}/\mathcal{S}\mapsto H^0(\mathrm{Spec}\,\mathbb{B}(\mathcal{X}), \mathcal{E}_{\mathcal{X}}|_{\mathrm{Spec}\,\mathbb{B}(\mathcal{X})}).$$

This is naturally a $\mathbb{B}$-module, thus, in particular an $E^{\diamond\mathrm{lf}}$-module. If W is an isocrystal, we write $\mathrm{BC}(W) \coloneqq \mathrm{BC}(\mathcal{E}(W))$, $\mathrm{BC}(W[1]) \coloneqq \mathrm{BC}(\mathcal{E}(W)[1])$, which lies over $\mathrm{Spd}\overline{\mathbb{F}}_q$, and $W\boxtimes\mathbb{B} \coloneqq \mathcal{E}(W)\boxtimes\mathbb{B}$, which lies over $\mathrm{Spd}\breve{E}$.

Working over $\mathrm{Spd}E$, if we consider the open immersion $j: \mathrm{Spec}\,\mathbb{B}_e\to\mathcal{X}$ and closed immersion $i: \mathcal{X}_{\square^{\sharp-\mathrm{alg}}}\hookrightarrow\mathcal{X}^{\mathrm{alg}}_{E,\square}$, then for any inscribed v-sheaf $\mathcal{S}/\mathrm{Spd}E$ and vector bundle $\mathcal{E}:\mathcal{S}\to\mathcal{X}^*\mathrm{Vect}$, there is a natural associated exact sequence of sheaves on $\mathcal{X}$ over $\mathcal{S}$

$$0\to\mathcal{E}_{\mathcal{X}}\to j_*j^*\mathcal{E}\to i_*\left((\mathcal{E}\boxtimes\mathbb{B}_{\mathrm{dR}})/(\mathcal{E}\boxtimes\mathbb{B}^{+}_{\mathrm{dR}})\right)\to 0 \tag{8.1.0.1}$$

Lemma 8.1.1. *Suppose $\mathcal{S}$ is an inscribed v-sheaf, and $\mathcal{E}:\mathcal{S}\to\mathcal{X}^*\mathrm{Vect}$. Then, each of $\mathrm{BC}(\mathcal{E})$ and $\mathrm{BC}(\mathcal{E}[1])$ is an inscribed v-sheaf. Moreover, if $\mathcal{S}/\mathrm{Spd}E$, then, $\mathcal{E}\boxtimes\mathbb{B}$ is an inscribed v-sheaf for $\mathbb{B}=\mathbb{B}_e,\mathbb{B}^{+}_{\mathrm{dR}}$, or $\mathbb{B}_{\mathrm{dR}}$, and the cohomology long exact sequences for Eq. (8.1.0.1) induce, by v-sheafification, an exact sequence of $E^{\diamond\mathrm{lf}}$-modules over $\mathcal{S}$*

$$0\to\mathrm{BC}(\mathcal{E})\to\mathcal{E}\boxtimes\mathbb{B}_e\to\mathcal{E}\boxtimes\mathbb{B}_{\mathrm{dR}}/\mathcal{E}\boxtimes\mathbb{B}^{+}_{\mathrm{dR}}\to\mathrm{BC}(\mathcal{E}[1])\to 0. \tag{8.1.1.1}$$

Proof. We have already seen in Section 6.3 that $\mathrm{BC}(\mathcal{E})$ and $\mathrm{BC}(\mathcal{E}[1])$ are inscribed v-sheaves, and that the $\mathcal{E}\boxtimes\mathbb{B}$ are inscribed v-sheaves follows, e.g., from Theorem 5.4.1 applied to the associated geometric vector bundles (or just from the property for $\mathbb{B}$ itself by writing $\mathcal{E}$ locally as a direct summand of $\mathbb{B}^n$). The exact sequence is then immediate from the definitions and the vanishing of quasi-coherent cohomology on affines. □

Definition 8.1.2. In the setting of Lemma 8.1.1, we refer to Eq. (8.1.1.1) as the *fundamental exact sequence* for $\mathcal{E}$.

8.2. **Automorphism groups.** Given an inscribed v-sheaf $\mathcal{S}$ and $\mathcal{E}:\mathcal{S}\to\mathcal{X}^*BG$, we write $\mathcal{G}_{\mathcal{E}}$ for the smooth affine scheme over $\mathcal{X}$ on $\mathcal{S}$ of automorphisms of $\mathcal{E}$. We write $\tilde{G}_{\mathcal{E}}$ for its moduli of sections, which is an inscribed v-sheaf by Theorem 5.4.1. If $\mathcal{S}/\mathrm{Spd}E$, then we can also form its moduli of sections $\mathcal{G}_{\mathcal{E}}(\mathbb{B}_e)$, $\mathcal{G}_{\mathcal{E}}(\mathbb{B}^+_{\mathrm{dR}})$, and $\mathcal{G}_{\mathcal{E}}(\mathbb{B}_{\mathrm{dR}})$, which are again inscribed v-stacks by Theorem 5.4.1. We note also that

$$\tilde{G}_{\mathcal{E}} = \mathcal{G}_{\mathcal{E}}(\mathbb{B}_e) \times_{\mathcal{G}_{\mathcal{E}}(\mathbb{B}_{\mathrm{dR}})} \mathcal{G}_{\mathcal{E}}(\mathbb{B}^+_{\mathrm{dR}}) =: \mathcal{G}_{\mathcal{E}}(\mathbb{B}_e) \cap \mathcal{G}_{\mathcal{E}}(\mathbb{B}^+_{\mathrm{dR}}). \tag{8.2.0.1}$$

Using the computation of the tangent bundle in Theorem 5.4.1, we find canonical identifications, compatible with restriction,

$$\mathrm{Lie}\,\tilde{G}_{\mathcal{E}} = \mathrm{BC}(\mathcal{E}(\mathfrak{g})) \text{ and } \mathrm{Lie}\,\mathcal{G}_{\mathcal{E}}(\mathbb{B}) = \mathcal{E}(\mathfrak{g})\boxtimes\mathbb{B} \text{ for } \mathbb{B} = \mathbb{B}_e, \mathbb{B}^+_{\mathrm{dR}}, \mathbb{B}_{\mathrm{dR}}. \tag{8.2.0.2}$$

8.3. An inscribed Hecke correspondence.

Lemma 8.3.1. *The natural map given by restriction of vector bundles*

$$\mathcal{X}^*\mathrm{Vect} \to (\mathrm{Spec}\,\mathbb{B}_e)^*\mathrm{Vect} \times_{(\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}})^*\mathrm{Vect}} (\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}})^*\mathrm{Vect}$$

is an equivalence of (inscribed) fibered categories.

Proof. Immediate from Beauville-Laszlo glueing. □

Proposition 8.3.2. *The natural map given by restriction of torsors*

$$\mathcal{X}^*\mathrm{B}G \to (\mathrm{Spec}\,\mathbb{B}_e)^*\mathrm{B}G \times_{(\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}})^*\mathrm{B}G} (\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}})^*\mathrm{B}G$$

is an equivalence of (inscribed) fibered categories.

Proof. Follows from Lemma 8.3.1 by the Tannakian formalism. □

Now, suppose given

$$\mathcal{E}_0 : \mathcal{S} \to \mathcal{X}^*\mathrm{B}G \text{ and a trivialization } \varphi_0 : \mathcal{E}_0|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}} \xrightarrow{\sim} \mathcal{E}_{\mathrm{triv}}.$$

Then, using Lemma 8.3.1, we obtain an induced map

$$m_{(\mathcal{E}_0,\varphi_0)} : \mathcal{S}\times_{\mathrm{Spd}\,E}\mathrm{Gr}_G \to \mathcal{X}^*\mathrm{B}G = (\mathrm{Spec}\,\mathbb{B}_e)^*\mathrm{B}G\times_{(\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}})^*\mathrm{B}G}(\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}})^*\mathrm{B}G \tag{8.3.2.1}$$

defined by

$$(s, (\mathcal{E}, \varphi : \mathcal{E}|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}} \xrightarrow{\sim} \mathcal{E}_{\mathrm{triv}})) \mapsto (\mathcal{E}_0|_{\mathrm{Spec}\,\mathbb{B}_e}, \mathcal{E}, \mathcal{E}_0|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}} \xrightarrow{\varphi^{-1}\circ\varphi_0} \mathcal{E}|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}}).$$

We write $\mathcal{G}_0 = \mathcal{G}_{\mathcal{E}_0}$ for the automorphism scheme of $\mathcal{E}_0$ as in Section 8.2. Note that restriction $\mathcal{G}_0(\mathbb{B}_e) \hookrightarrow \mathcal{G}_0(\mathbb{B}_{\mathrm{dR}})$ followed by conjugation by φ_0 (an isomorphism $\mathcal{G}_0(\mathbb{B}_{\mathrm{dR}}) \xrightarrow{\sim} G(\mathbb{B}_{\mathrm{dR}})$) induces a map $\mathcal{G}_0(\mathbb{B}_e) \hookrightarrow G(\mathbb{B}_{\mathrm{dR}})$.

Lemma 8.3.3. *The map $p_1 \times m_{\mathcal{E}_0,\varphi_0} : \mathcal{S} \times \mathrm{Gr}_G \to \mathcal{S} \times \mathcal{X}^*\mathrm{B}G$ of inscribed v-stacks over $\mathcal{S}$ is a quasi-torsor for the action of $\mathcal{G}_0(\mathbb{B}_e) \leq G(\mathbb{B}_{\mathrm{dR}})$.*

Proof. Suppose $(s, (\mathcal{E}, \varphi))$ and $(s, (\mathcal{E}', \varphi'))$ map to the same object. Then $\mathcal{E}$ is isomorphic to $\mathcal{E}'$, so we may assume $\mathcal{E} = \mathcal{E}'$. It then follows that there is an automorphism ψ of $(\mathcal{E}_0|_{\mathrm{Spec}\,\mathbb{B}_e})$ such that

$$\varphi^{-1} \circ \varphi_0 \circ \psi = (\varphi')^{-1} \circ \varphi_0.$$

and thus

$$\varphi^{-1} \circ (\varphi_0 \circ \psi \circ \varphi_0^{-1}) = (\varphi')^{-1}.$$

Thus $\varphi_0 \circ \psi \circ \varphi_0^{-1}$ gives an element of $\mathcal{G}_0(\mathbb{B}_e) \subseteq G(\mathbb{B}_{\mathrm{dR}})$ mapping the one pre-image to the other. Reversing the argument we find that if there is

$g \in G(\mathbb{B}_{\mathrm{dR}})$ such that $(s,(\mathcal{E},g\varphi))$ and $(S,(\mathcal{E},\varphi))$ map to the same object, then $g \in \mathcal{G}_0(\mathbb{B}_e)$. □

8.4. **Generalized Newton strata on the $\mathbb{B}^+_{\mathrm{dR}}$-affine Grassmannian.** Let G/E be a connected linear algebraic and let b_1 in $\mathrm{Spd}\breve{E}$. We pullback the bundle $\mathcal{E}_{b_1}$ of Section 6.4 from $\mathrm{Spd}\overline{\mathbb{F}}_q$ to $\mathrm{Spd}\breve{E}$.

By construction, there is a canonical trivialization $\varphi_{b_1} : \mathcal{E}_{b_1}|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}} \xrightarrow{\sim} \mathcal{E}_{\mathrm{triv}}$. By the construction of Section 8.3, we obtain an induced map

$$m = m_{b_1,\varphi_{b_1}} : \mathrm{Spd}\breve{E} \times_{\mathrm{Spd}E} \mathrm{Gr}_G \to \mathcal{X}^*\mathrm{B}G.$$

For $[b_2] \in B(G)$, the set of σ-conjugacy classes in $G(\breve{E})$, we write

$$\mathrm{Gr}_G^{b_1\to[b_2]} := m \times_{\mathcal{X}^*\mathrm{B}G} ((\mathcal{X}^*\mathrm{B}G)^{[b_2]} \to \mathcal{X}^*\mathrm{B}G).$$

If we fix also $[\mu]$ a conjugacy class of cocharacters of $G_{\overline{L}}$, we write

$$\mathrm{Gr}_{[\mu]}^{b_1\to[b_2]} = (\mathrm{Gr}_{[\mu]} \times_{\mathrm{Spd}E([\mu])} \mathrm{Spd}\breve{E}([\mu])) \times_{\mathrm{Gr}_G \times_{\mathrm{Spd}E}\mathrm{Spd}\breve{E}} \mathrm{Gr}_G^{b_1\to[b_2]}.$$

Example 8.4.1. By Remark 6.4.2 and the results of [7], when G is reductive, the $(\mathrm{Gr}_G^{b_1\to[b_2]})_0$ as $[b]$ varies give a stratification of $(\mathrm{Gr}_G \times_{\mathrm{Spd}E} \mathrm{Spd}\breve{E})_0$ by locally closed subsheaves. For $b_1 = 1$, this is called the Newton stratification.

These stratifications are usually studied after restricting to the Schubert cells $\mathrm{Gr}_{[\mu]}$. In particular, for a conjugacy class $[\mu]$, let $b_1 \in B(G,[\mu])$, the Kottwitz set . Then $(\mathrm{Gr}_{[\mu]}^{b_1\to[1]})_0$ is the open non-empty admissible locus for b_1 (with respect to $[\mu]$), and the other non-empty terms $(\mathrm{Gr}_{[\mu]}^{b_1\to[b_2]})_0$ stratify the boundary.

Remark 8.4.2. If we fix $b \in [b]$, then we obtain a canonical $\tilde{G}_b := \tilde{G}_{\mathcal{E}_b}$-torsor

$$(m_{b_1,\varphi_{b_1}} \times \mathrm{Id}_{\mathrm{Spd}\breve{E}}) \times_{\mathcal{X}^*\mathrm{B}G\times\mathrm{Spd}\breve{E}} (\mathcal{E}_b \times \mathrm{Id}_{\mathrm{Spd}\breve{E}}) \to \mathrm{Gr}_G^{b_0\to[b]}.$$

parameterizing trivializations of the modified bundle to $\mathcal{E}_b$. This torsor admits a natural equivariant action of $\mathcal{G}_{b_0}(\mathbb{B}_e) := \mathcal{G}_{\mathcal{E}_{b_0}}(\mathbb{B}_e)$ and it follows from Lemma 8.3.3 that this action realizes the structure map to $\mathrm{Spd}\breve{E}$ as a quasi-torsor. We will see in Theorem 9.1.5 that this is in fact a torsor (i.e., it is surjective or equivalently in this case non-empty) and admits a simpler description that highlights the symmetry between b_1 and b_2. Using this description, we will then deduce an explicit computation of the tangent and normal bundles of $\mathrm{Gr}_G^{b_1\to[b_2]}$ (Corollary 9.1.7) and $\mathrm{Gr}_{[\mu]}^{b_1\to[b_2]}$ (Corollary 9.2.4).

9. Moduli of modifications

In this section we define our moduli of modifications and establish their main properties. We first treat the unbounded moduli space in Section 9.1. Its main structures are described in Theorem 9.1.5, including the key description as a bitorsor over $\mathrm{Spd}\breve{E}$. In Corollary 9.1.6 we then deduce a computation of its tangent bundle and the derivatives of its natural period maps and a computation of the tangent and normal bundles of the associated generalized Newton strata Corollary 9.1.7. In Section 9.2 we then cut out the bounded moduli space inside by taking the preimage of a Schubert cell under a period map. Combining the results on the unbounded moduli space of Section 9.1 and the results on the B^+_{dR}-affine Grassmannian and its Schubert cells of Section 7, we obtain in Theorem 9.2.1 (generalizing Theorem B) a description of the main structures of the bounded moduli of modifications, in Corollary 9.2.3 (generalizing Theorem C) a description of its tangent bundle and the derivatives of its period maps, and in Corollary 9.2.4 a description of the tangent and normal bundles of the associated generalized Newton strata. Finally, in Section 9.3, we describe a very general two towers isomorphism for inscribed moduli of modifications and explain how it interacts with our computations of tangent bundles and derivatives.

9.0.1. *Notation.* We fix a finite extension $E/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. In this section we work in the inscribed context $(\mathrm{Spd}\mathbb{F}_q, X_{E,\square})$, and use freely the notation of Section 8. We also use the notation of Section 7, transported into this inscribed setting by change of context as in Section 5.5. In particular, we view Gr_G as an inscribed v-sheaf on $X^{\mathrm{lf}}_{E,\square}$ over $\mathrm{Spd}E$, i.e. by

$$\mathrm{Gr}_G(\mathcal{X}/X^{\mathrm{alg}}_{E,P}) = \{(P/\mathrm{Spd}E, s : \mathcal{X}|_{P^{\sharp-\mathrm{alg}}_{(\infty)}} \to \mathrm{Gr}_G)\},$$

and similarly for the Schubert cells $\mathrm{Gr}_{[\mu]}$, etc.

9.1. **The unbounded moduli space.** Recall from Section 6.4 that, for G/E a connected linear algebraic group and any $b \in G(\breve{E})$, we have defined a G-bundle $\mathcal{E}_b : \mathrm{Spd}\overline{\mathbb{F}}_q \to \mathcal{X}^*\mathrm{B}G$. We write $\mathcal{G}_b = \mathcal{G}_{\mathcal{E}_b}$ for the automorphism scheme of $\mathcal{E}_b$ as in Section 8.2 and $\tilde{G}_b = \tilde{G}_{\mathcal{E}_b}$ for its moduli of global sections.

After restriction to $\mathrm{Spd}\breve{E} = \mathrm{Spd}E \times_{\mathrm{Spd}\mathbb{F}_q} \mathrm{Spd}\overline{\mathbb{F}}_q$, there is a canonical trivialization

$$\mathrm{triv}_b : \mathcal{E}_b|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}} \to \mathcal{E}_{\mathrm{triv}}.$$

In the remainder of this subsection, all inscribed v-sheaves have been base changed to lie over $\mathrm{Spd}\breve{E}$ (with its trivial inscription). Equivalently, as described in Section 5.5, we could work in the inscribed context $(\mathrm{Spd}\breve{E}, X_{E,\square})$.

Definition 9.1.1. Let G/E be a connected linear algebraic group, and let $b_1, b_2 \in G(\breve{E})$. We define $\mathscr{M}_{b_1\to b_2}$ to be the presheaf on $X^{\mathrm{lf}}_{E,\square}$ over $\mathrm{Spd}\breve{E}$:

$$\mathscr{M}_{b_1\to b_2} = (\mathrm{Spd}\breve{E} \xrightarrow{\mathcal{E}_{b_1}|_{\mathcal{X}\backslash\infty}} (\mathcal{X}\backslash\infty)^*\mathrm{B}G) \times_{(\mathcal{X}\backslash\infty)^*\mathrm{B}G} (\mathrm{Spd}\breve{E} \xrightarrow{\mathcal{E}_{b_2}|_{\mathcal{X}\backslash\infty}} (\mathcal{X}\backslash\infty)^*\mathrm{B}G).$$

Equivalently, $\mathscr{M}_{b_1\to b_2}$ is the functor

$$(\mathcal{X}/X^{\mathrm{alg}}_{E,P}, P/\mathrm{Spd}\breve{E}) \mapsto \{\varphi : \mathcal{E}_{b_1}|_{\mathcal{X}\backslash\infty} \xrightarrow{\sim} \mathcal{E}_{b_2}|_{\mathcal{X}\backslash\infty}\}.$$

It follows from Theorem 6.1.2 that $\mathscr{M}_{b_1\to b_2}$ is an inscribed v-sheaf over $\mathrm{Spd}\breve{E}$, and it admits obvious actions of $\mathcal{G}_{b_i}(\mathbb{B}_e)$, $i = 1, 2$, by precomposition and postcomposition with φ. Explicitly, we define the right action maps

$$a_i : \mathcal{M}_{b_1\to b_2} \times_{\mathrm{Spd}\breve{E}} \mathcal{G}_{b_i}(\mathbb{B}_e) \to \mathcal{M}_{b_1\to b_2}, i = 1,2, \text{ by}$$
$$a_1(\varphi, g) = \varphi \circ g \text{ and } a_2(\varphi, g) = g^{-1} \circ \varphi.$$

Remark 9.1.2. Unwinding the definitions and proof of Theorem 6.1.2, we see that $\mathcal{M}_{b_1\to b_2}$ is the moduli of sections as in Theorem 5.4.1 for the affine scheme $\mathcal{G}_{b_1\to b_2}$ over $\mathcal{X}$ on $\mathrm{Spd}\breve{E}$ of isomorphisms of G-torsors, $\mathcal{I}som(\mathcal{E}_{b_1}, \mathcal{E}_{b_2})$, and the actions of $\mathcal{G}_{b_i}(\mathbb{B}_e)$ are induced by the actions of $\mathcal{G}_{b_i}$ on this scheme.

We also define $c_{\mathrm{dR}} : \mathcal{M}_{b_1\to b_2} \to G(\mathbb{B}_{\mathrm{dR}})$ to be the map

$$\varphi \mapsto \mathrm{triv}_{b_2}^{-1} \circ \varphi|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}} \circ \mathrm{triv}_{b_1}.$$

We define period maps $\pi_i : \mathcal{M}_{b_1\to b_2} \to \mathrm{Gr}_G$ by

$$\pi_1(\varphi) = (\mathcal{E}_{b_1}|_{\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}}, \mathrm{triv}_{b_2}^{-1} \circ \varphi|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}}) \text{ and}$$
$$\pi_2(\varphi) = (\mathcal{E}_{b_2}|_{\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}}, \mathrm{triv}_{b_1}^{-1} \circ \varphi^{-1}|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}}).$$

Lemma 9.1.3. *The maps $\pi_i : \mathcal{M}_{b_1\to b_2} \to \mathrm{Gr}_G$ are computed via c_{dR} as*

$$\pi_1(\varphi) = c_{\mathrm{dR}}(\varphi) \cdot *_1 \text{ and } \pi_2(\varphi) = (c_{\mathrm{dR}}(\varphi))^{-1} \cdot *_1.$$

Proof. For π_1, the map triv_{b_2} gives an isomorphism

$$c_{\mathrm{dR}}(\varphi) \cdot *_1 = (\mathcal{E}_1|_{\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}}, c_{\mathrm{dR}}(\varphi)) \xrightarrow{\sim} (\mathcal{E}_{b_2}|_{\mathrm{Spec}\,\mathbb{B}^+_{\mathrm{dR}}}, \varphi|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}} \circ \mathrm{triv}_{b_1})$$

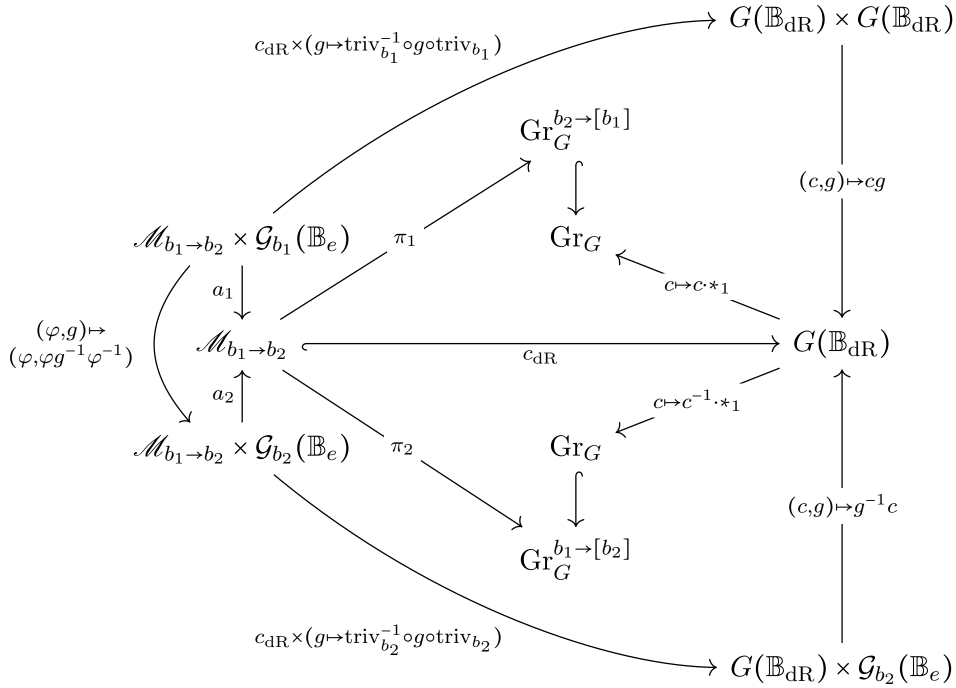
The argument is similar for π_2. □

Example 9.1.4. For $b \in G(\breve{E})$, the functor $\mathscr{M}_b$ of the introduction is $\mathscr{M}_{1\to b}$, and the maps a_i, c_{dR}, and π_i defined here specialize in this case to the maps considered in the introduction.

Theorem 9.1.5. *Let G/E be a connected linear algebraic group, let $b_1, b_2 \in G(\breve{E})$. Then $\mathscr{M}_{b_1\to b_2}$ is an inscribed v-sheaf over $\mathrm{Spd}\breve{E}$. Moreover,*

(1) *The action maps a_1 and a_2 realize $\mathscr{M}_{b_1\to b_2}$ as a bitorsor over $\mathrm{Spd}\breve{E}$, trivializeable over $\mathrm{Spd}\mathbb{C}_p$.*
(2) *The map π_1 factors through $\mathrm{Gr}_G^{b_2\to[b_1]}$ and the map π_2 factors through $\mathrm{Gr}_G^{b_1\to[b_2]}$. The restriction of the action map a_1 to $\tilde{G}_{b_1}$ realizes π_1 as the canonical $\mathcal{G}_{b_2}(\mathbb{B}_e)$-equivariant $\tilde{G}_{b_1}$-torsor over $\mathrm{Gr}_G^{b_2\to[b_1]}$ of Remark 8.4.2 and the restriction of the action map a_2 to $\tilde{G}_{b_2}$ realizes π_2 as the canonical $\mathcal{G}_{b_1}(\mathbb{B}_e)$-equivariant $\tilde{G}_{b_2}$-torsor over $\mathrm{Gr}_G^{b_1\to[b_2]}$ of Remark 8.4.2.*

and the following diagram commutes:



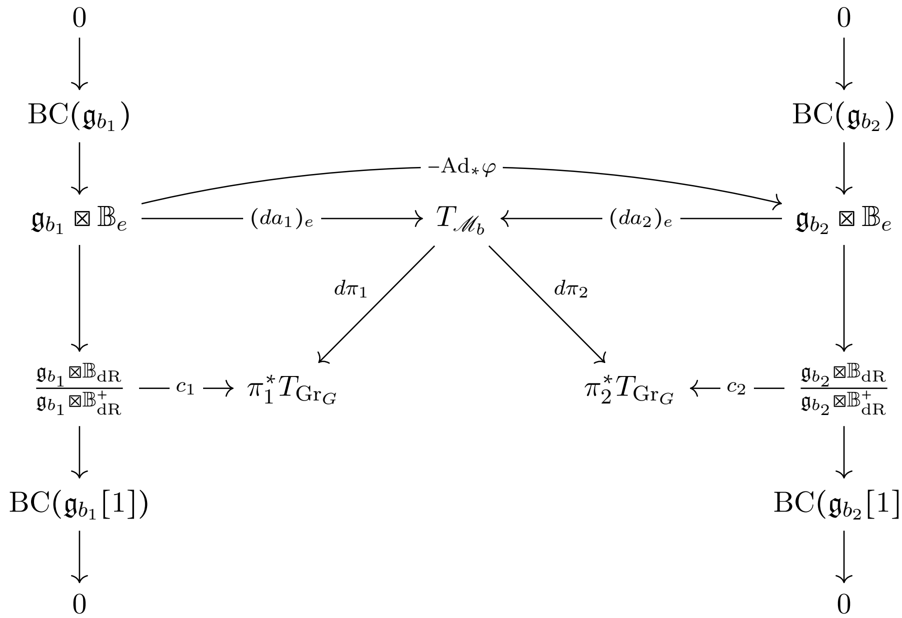
Proof. It is an inscribed v-sheaf by Theorem 6.1.2. It is evidently a quasi-bitorsor for $\mathcal{G}_{b_1}(\mathbb{B}_e)$ and $\mathcal{G}_{b_2}(\mathbb{B}_e)$. It is trivialized over $\mathrm{Spd}\mathbb{C}_p$ (and thus, in particular, a bitorsor) by [1, Theorem 6.5] (to apply this result in the case of G non-reductive, we note that any b may be σ-conjugated into a Levi subgroup of G). The commutativity of the diagram is a chase through the definitions after applying Lemma 9.1.3. □

We now describe the differentials of the maps in the commutative diagram of Theorem 9.1.5. To that end, note that, writing $\mathfrak{g}_{b_i}$ for the isocrystal associated to b_i by the adjoint representation on $\mathfrak{g}$, we have $\pi_i^* \mathfrak{g}^+_{\mathrm{univ}} = \mathfrak{g}_{b_i} \boxtimes \mathbb{B}^+_{\mathrm{dR}}$. Then, from Corollary 7.1.6, we obtain canonical isomorphisms

$$c_i : (\mathfrak{g}_{b_i} \boxtimes \mathbb{B}_{\mathrm{dR}})/(\mathfrak{g}_{b_i} \boxtimes \mathbb{B}^+_{\mathrm{dR}}) \xrightarrow{\sim} \pi_i^* T\mathrm{Gr}_G. \tag{9.1.5.1}$$

Corollary 9.1.6. *The following diagram of $E^{\diamond \mathrm{lf}}$-modules on $\mathscr{M}_{b_1 \to b_2}$ commutes, where the left and right columns are the fundamental exact sequences of Lemma 8.1.1 for $\mathfrak{g}_{b_1}$ and $\mathfrak{g}_{b_2}$ and the morphisms c_i are as in Eq. (9.1.5.1).*

Moreover, the horizontal arrows are all isomorphisms.



Proof. It follows from the torsor property that each of $(da_i)_e$ is an isomorphism, and the identity $(da_1)_e = -(da_2)_e \circ \mathrm{Ad}_*\varphi$ is immediate from the commutative circle at the top of the diagram in Theorem 9.1.5.

The commutativity of the left middle quadrilateral comes from using Theorem 9.1.5 to compute $d\pi_1 \circ da_1$ as the derivative of

$$\begin{aligned}(\varphi, g) &\mapsto (c_{\mathrm{dR}}(\varphi)\mathrm{triv}_{b_1}^{-1} g \mathrm{triv}_{b_1} c_{\mathrm{dR}}(\varphi)^{-1}) c_{\mathrm{dR}}(\varphi) \cdot *_1 \\ &= \big((\mathrm{triv}_{b_2}^{-1} \circ \varphi) g (\mathrm{triv}_{b_2}^{-1} \circ \varphi)^{-1}\big) \cdot (c_{\mathrm{dR}}(\varphi) \cdot *_1)\end{aligned}$$

Indeed, the identification c_1 is given by composing the derivative of the action map $G(\mathbb{B}_{\mathrm{dR}}) \times_{\mathrm{Spd}\breve{E}} \mathrm{Gr}_G \to \mathrm{Gr}_G$ at the identity in $G(\mathbb{B}_{\mathrm{dR}})$ with the isomorphism $\mathfrak{g}_{b_1} \otimes \mathbb{B}_{\mathrm{dR}} \to \mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}}$ induced by $\mathrm{triv}_{b_2}^{-1} \circ \varphi|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}}$.

Similarly, for commutativity of the right middle quadrilateral we compute $d\pi_2 \circ da_2$ as the derivative of

$$\begin{aligned}(\varphi, g) &\mapsto (c_{\mathrm{dR}}(\varphi)^{-1}\mathrm{triv}_{b_2}^{-1} g \mathrm{triv}_{b_2} c_{\mathrm{dR}}(\varphi)) c_{\mathrm{dR}}(\varphi)^{-1} \cdot *_1 \\ &= \big((\mathrm{triv}_{b_1}^{-1} \circ \varphi^{-1}) g (\mathrm{triv}_{b_1}^{-1} \circ \varphi^{-1})^{-1}\big) \cdot (c_{\mathrm{dR}}(\varphi)^{-1} \cdot *_1)\end{aligned}$$

where we note the inverses in the definitions of a_2 and π_2 are cancelling to give the term g. Indeed, the identification c_2 is given by composing the derivative of the action map $G(\mathbb{B}_{\mathrm{dR}}) \times_{\mathrm{Spd}\breve{E}} \mathrm{Gr}_G \to \mathrm{Gr}_G$ at the identity in $G(\mathbb{B}_{\mathrm{dR}})$ with the isomorphism $\mathfrak{g}_{b_2} \otimes \mathbb{B}_{\mathrm{dR}} \to \mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}}$ induced by $\mathrm{triv}_{b_1}^{-1} \circ \varphi^{-1}|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}}$. We obtain the negative sign in the commutative diagram because of the g^{-1} that appears instead of a g within the conjugation. □

Corollary 9.1.7. *Let $\mathcal{E}_{[b_2]}$ be the restriction of the universal G-bundle on $\mathcal{X}^*\mathrm{B}G$ to $(\mathcal{X}^*\mathrm{B}G)^{[b_2]}$ and let $\mathfrak{g}_{[b_2]} = \mathcal{E}_{[b_2]}(\mathfrak{g})$ be its push-out by the adjoint representation. The short exact sequence over $\mathrm{Gr}_G^{b_1\to[b_2]}$ induced by the fundamental exact sequence of Lemma 8.1.1 for $\mathfrak{g}_{[b_2]}$,*

$$0 \to \mathfrak{g}_{[b_2]} \boxtimes \mathbb{B}_e/\mathrm{BC}(\mathfrak{g}_{[b_2]}) \to \mathfrak{g}_{[b]} \boxtimes \mathbb{B}_{\mathrm{dR}}/\mathfrak{g}_{[b_2]} \boxtimes \mathbb{B}^+_{\mathrm{dR}} \to \mathrm{BC}(\mathfrak{g}_{[b_2]}[1]) \to 0,$$

is canonically identified with the short exact sequence

$$0 \to T_{\mathrm{Gr}_G^{b_1\to[b_2]}} \xrightarrow{d\iota} \iota^* T_{\mathrm{Gr}_G} \to N_{\mathrm{Gr}_G^{b_1\to[b_2]}} \to 0$$

where $\iota : \mathrm{Gr}_G^{b_1\to[b_2]} \hookrightarrow \mathrm{Gr}_G$ is the inclusion.

Remark 9.1.8. The situation is symmetric in b_1 and b_2, so we only need to state one version in Corollary 9.1.7.

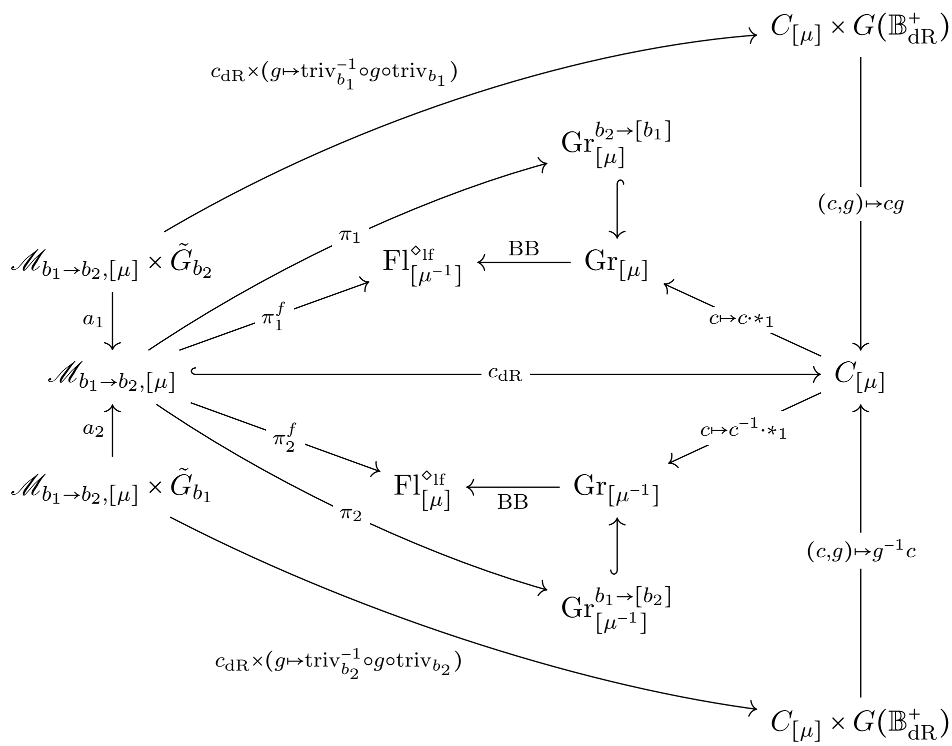
9.2. **The bounded moduli space.** We fix G/E a connected linear algebraic group and $[\mu]$ a conjugacy class of cocharacters of $G_{\overline{E}}$. In this subsection all objects are base changed to $\mathrm{Spd}(\breve{E}([\mu]))$. For $b_1, b_2 \in G(\breve{E})$,

$$\mathscr{M}_{b_1\to b_2,[\mu]} := c_{\mathrm{dR}} \times_{G(\mathbb{B}_{\mathrm{dR}})} C_{[\mu]} = \pi_1 \times_{\mathrm{Gr}_G} \mathrm{Gr}_{[\mu]} = \pi_2 \times_{\mathrm{Gr}_G} \mathrm{Gr}_{[\mu^{-1}]}$$

where the second two equalities are immediate from the definitions. The left and right multipliation actions of $G(\mathbb{B}_{\mathrm{dR}})$ on itself restrict to left and right multiplication actions of $G(\mathbb{B}^+_{\mathrm{dR}})$ on $C_{[\mu]}$, thus by Eq. (8.2.0.1), the actions of $\mathcal{G}_{b_i}(\mathbb{B}_e)$ on $\mathscr{M}_{b_1\to b_2}$ restrict to actions of $\tilde{G}_{b_i}$ on $\mathscr{M}_{b_1\to b_2,[\mu]}$. Below we write a_i for the restrictions of the right action maps used in Section 9.1.

Below we also write π_i for the restriction of the period maps to $\mathscr{M}_{b_1\to b_2,[\mu]}$, so that π_1 factors through $\mathrm{Gr}_{[\mu]}$ and π_2 factors through $\mathrm{Gr}_{[\mu^{-1}]}$. We then obtain two filtration period maps write $\pi_1^f = \mathrm{BB} \circ \pi_1 : \mathscr{M}_{b_1\to b_2,[\mu]} \to \mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu^{-1}]}$ and $\pi_2^f = \mathrm{BB} \circ \pi_2 : \mathscr{M}_{b_1\to b_2,[\mu]} \to \mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu]}$.

Theorem 9.2.1. *Let $b_1, b_2 \in G(\breve{E})$. Then $\mathscr{M}_{b_1\to b_2,[\mu]}$ is a inscribed v-sheaf over $\mathrm{Spd}\breve{E}([\mu])$. The map π_1 factors through $\mathrm{Gr}_{[\mu]}^{b_2\to[b_1]}$ and the map π_2 factors through $\mathrm{Gr}_{[\mu]}^{b_1\to[b_2]}$, and the actions realizes π_1 as the canonical $\tilde{G}_{b_2}$-equivariant $\tilde{G}_{b_1}$-torsor over $\mathrm{Gr}_{[\mu]}^{b_2\to[b_1]}$ of Remark 8.4.2 and π_2 as the canonical $\tilde{G}_{b_1}$-equivariant $\tilde{G}_{b_2}$-torsor over $\mathrm{Gr}_{[\mu^{-1}]}^{b_1\to[b_2]}$ of Remark 8.4.2. The following* *extended subdiagram of the diagram in Theorem 9.1.5 commutes:*



Proof. That $\mathscr{M}_{b_1 \to b_2,[\mu]}$ is an inscribed v-sheaf follows from the corresponding property of the constituents of the fiber product and Lemma 4.1.7. The rest of the theorem follows by restriction from Theorem 9.2.1. □

Remark 9.2.2. Note that $\mathcal{M}_{b_1 \to b_2,[\mu]}$ may be empty. When $b_1 = 1$, it is non-empty exactly when $[b_2]$ lies in the Kottwitz set $B(G,[\mu^{-1}])$.

As in the unbounded case, we can now describe the derivatives. To that end, we need to consider some bounded analogs of the fundamental exact sequence. We write $\mathcal{E}_{\max}$ for the minimal common modification of $\mathcal{E}(\mathfrak{g}_1)$ and $\mathcal{E}(\mathfrak{g}_2)$ on $\mathcal{M}_{b_1 \to b_2,[\mu]}$, i.e. for the modification associated by Lemma 8.3.1 to

$$\mathfrak{g}_1 \boxtimes \mathbb{B}_{\mathrm{dR}}^+ + \mathfrak{g}_2 \boxtimes \mathbb{B}_{\mathrm{dR}}^+ = \pi_1^* \mathfrak{g}_{\max}^+ = \pi_2^* \mathfrak{g}_{\max}^+,$$

which is a lattice in $\mathfrak{g}_1 \boxtimes \mathbb{B}_{\mathrm{dR}} = \mathfrak{g}_2 \boxtimes \mathbb{B}_{\mathrm{dR}}$ by Proposition 7.2.3.

We write this lattice as $\mathfrak{g}_{\max}^+$. Then, for each i, we have an exact sequence of sheaves on $\mathcal{X}$ over $\mathcal{M}_{b_1 \to b_2,[\mu]}$:

$$0 \to \mathcal{E}(\mathfrak{g}_{b_i}) \to \mathcal{E}_{\max} \to \iota_*(\mathfrak{g}_{\max}^+ / \mathfrak{g}_i \boxtimes \mathbb{B}_{\mathrm{dR}}^+) \to 0. \tag{9.2.2.1}$$

The v-sheafification of the associated long exact sequence of cohomology gives rise to an exact sequence of $E^{\diamond \mathrm{lf}}$-modules on $\mathscr{M}_{b_1 \to b_2,[\mu]}$,

(9.2.2.2)

$$0 \to \mathrm{BC}(\mathfrak{g}_{b_i}) \to \mathrm{BC}(\mathcal{E}_{\max}) \to \frac{\mathfrak{g}_{\max}^+}{\mathfrak{g}_i \boxtimes \mathbb{B}_{\mathrm{dR}}^+} \to \mathrm{BC}(\mathfrak{g}_{b_i}[1]) \to \mathrm{BC}(\mathcal{E}_{\max}[1]) \to 0$$

where the last zero is simply because the third term in Eq. (9.2.2.1) is a quasi-coherent sheaf supported on a closed affine subscheme of $\mathcal{X}$ so has vanishing cohomology.

Corollary 9.2.3. *The following diagram of inscribed $E^{\diamond\mathrm{lf}}$-modules on $\mathscr{M}_{b_1\to b_2,[\mu]}$, which is an extended subdiagram of (the pullback to $\mathscr{M}_{b_1\to b_2,[\mu]}$ of) the diagram of Corollary 9.1.6, commutes. The left and right columns are the bounded fundamental exact sequences of Eq. (9.2.2.2) for $\mathfrak{g}_{b_1}$ and $\mathfrak{g}_{b_2}$, and the morphisms c_i are obtained by restricting Eq. (9.1.5.1). Moreover, the horizontal arrows are all isomorphisms.*

$$
\begin{array}{ccccccc}
0 & & & & & & 0 \\
\downarrow & & & & & & \downarrow \\
\mathrm{BC}(\mathfrak{g}_{b_1}) & & & & & & \mathrm{BC}(\mathfrak{g}_{b_2}) \\
\downarrow & & & \xrightarrow{\quad -\mathrm{Ad}_*\varphi \quad} & & & \downarrow \\
\mathrm{BC}(\mathcal{E}_{\max}) & & \xrightarrow{(da_1)_e} & T_{\mathscr{M}_{b_1\to b_2,[\mu]}} & \xleftarrow{(da_2)_e} & & \mathrm{BC}(\mathcal{E}_{\max}) \\
\downarrow & & \swarrow{\scriptstyle d\pi_1} & & \searrow{\scriptstyle d\pi_2} & & \downarrow \\
\frac{\mathfrak{g}^+_{\max}}{\mathfrak{g}_{b_1}\boxtimes\mathbb{B}^+_{\mathrm{dR}}} & \xrightarrow{c_1} & \pi_1^* T_{\mathrm{Gr}_{[\mu]}} & & \pi_2^* T_{\mathrm{Gr}_{[\mu^{-1}]}} & \xleftarrow{c_2} & \frac{\mathfrak{g}^+_{\max}}{\mathfrak{g}_{b_2}\boxtimes\mathbb{B}^+_{\mathrm{dR}}} \\
\downarrow & & & & & & \downarrow \\
\mathrm{BC}(\mathfrak{g}_{b_1}[1]) & & & & & & \mathrm{BC}(\mathfrak{g}_{b_2}[1]) \\
\downarrow & & & & & & \downarrow \\
\mathrm{BC}(\mathcal{E}_{\max}[1]) & & & & & & \mathrm{BC}(\mathcal{E}_{\max}[1]) \\
\downarrow & & & & & & \downarrow \\
0 & & & & & & 0
\end{array}
$$

Finally, we obtain also a description of the tangent bundle and normal bundle for the bounded generalized Newton strata.

Corollary 9.2.4. *Let $\mathcal{E}_{[b_2]}$ be the restriction of the universal G-bundle on $\mathcal{X}^*\mathrm{B}G$ to $(\mathcal{X}^*\mathrm{B}G)^{[b_2]}$ and let $\mathfrak{g}_{[b_2]} = \mathcal{E}_{[b_2]}(\mathfrak{g})$ be its push-out by the adjoint representation. The short exact sequence over $\mathrm{Gr}^{b_1\to[b_2]}_{[\mu]}$ induced by the bounded fundamental exact sequence analogous to Eq. (9.2.2.2) for $\mathfrak{g}_{[b_2]}$,*

$$0 \to \frac{\mathrm{BC}(\mathcal{E}_{\max})}{\mathrm{BC}(\mathfrak{g}_{[b_2]})} \to \frac{\mathfrak{g}^+_{\max}}{\mathfrak{g}_{[b_2]}\boxtimes\mathbb{B}^+_{\mathrm{dR}}} \to \mathrm{Ker}\left(\mathrm{BC}(\mathfrak{g}_{[b_2]}[1]) \to \mathrm{BC}(\mathcal{E}_{\max}[1])\right) \to 0,$$

is canonically identified with the short exact sequence

$$0 \to T_{\mathrm{Gr}^{b_1\to[b_2]}_{[\mu]}} \xrightarrow{d\iota} \iota^* T_{\mathrm{Gr}_G} \to N_{\mathrm{Gr}^{b_1\to[b_2]}_{[\mu]}} \to 0$$

where $\iota : \mathrm{Gr}_{[\mu]}^{b_1\to[b_2]} \hookrightarrow \mathrm{Gr}_{[\mu]}$ *is the inclusion. This isomorphism is moreover compatible with that of Corollary 9.1.7 by the natural inclusion maps.*

9.3. **The two towers.** There is an evident isomorphism

$$\mathscr{M}_{b_1\to b_2} \xrightarrow{\sim} \mathscr{M}_{b_2\to b_1}, \varphi \mapsto \varphi^{-1}.$$

This isomorphism reflects the diagrams of Theorem 9.1.5 and Theorem 9.2.1 along the central horizontal axis, acting as $c \mapsto c^{-1}$ on $G(\mathbb{B}_{\mathrm{dR}})$ (and thus it reflects the diagrams of Corollary 9.1.6 and Corollary 9.2.3 along the central vertical axis).

One also obtains interesting isomorphisms by changing the group; combining these observations will give the traditional two towers isomorphism (in the form discussed, e.g., in [13, §8.5]) in a very general setting. To state this cleanly, it is useful to consider the following generalization of our constructions:

Definition 9.3.1. For G/E a connected linear algebraic group, an affine group scheme $\mathcal{G}$ on $\mathcal{X}$ over $\mathrm{Spd}\breve{E}$ is an $\mathcal{X}$-pure inner form of G if it is isomorphic to the automorphism group scheme $\mathcal{A}ut_{\mathcal{X}}(\mathcal{E})$ of a G-torsor $\mathcal{E} : \mathrm{Spd}\breve{E} \to \mathcal{X}^*\mathrm{B}G$.

Example 9.3.2. For $b \in G(\breve{E})$, $\mathcal{A}ut(\mathcal{E})$ is the affine group scheme we have denoted by $\mathcal{G}_b$ above. If b is basic, then $\mathcal{G}_b = G_b \times_E \mathcal{X}$ where G_b is the automorphism group of the G-isocrystal b.

Given $\mathcal{G}$ that is an $\mathcal{X}$-pure inner form of a connecetd linear algebraic group G/E, if we fix a G-bundle $\mathcal{E}$ and an isomorphism $\mathcal{G} = \mathcal{A}ut(\mathcal{E})$, we obtain a twisting isomorphism

$$\mathcal{X}^*\mathrm{B}G \xrightarrow{\sim} \mathcal{X}^*\mathrm{B}\mathcal{G}, \mathcal{E}' \mapsto \mathcal{I}som_G(\mathcal{E}, \mathcal{E}').$$

In particular, it follows that $\mathcal{X}^*\mathrm{B}\mathcal{G}$ is an inscribed v-stack.

Suppose now given such a $\mathcal{G}$, and $\mathcal{G}$-torsors $\mathcal{E}_1, \mathcal{E}_2 : \mathrm{Spd}\breve{E} \to \mathcal{X}^*\mathrm{B}\mathcal{G}$ equipped with trivializations $\mathrm{triv}_i : \mathcal{E}_i|_{\mathrm{Spec}\,\mathbb{B}_{\mathrm{dR}}} \xrightarrow{\sim} \mathcal{E}_{\mathrm{triv}}$. Then, we can define the moduli of modifications $\mathcal{M}_{\mathcal{E}_1\to\mathcal{E}_2}$, the maps c_{dR} and π_i, the actions of automorphism groups, etc., by imitating the discussion given above. In particular, one again obtains an isomorphism $\mathscr{M}_{\mathcal{E}_1\to\mathcal{E}_2}$ that acts as $c \mapsto c^{-1}$ on $G(\mathbb{B}_{\mathrm{dR}})$ and reflects the same diagrams.

Combining these two constructions, one obtains (an extension of) the classical two towers duality for the infinite level moduli space $\mathscr{M}_{1\to b}$: first, we apply $\mathcal{I}som_G(\mathcal{E}_b, \bullet)$ to obtain an isomorphism with $\mathscr{M}_{\mathcal{E}'\to\mathcal{E}'_{\mathrm{triv}}}$ where in the subscript we have the $\mathcal{G}_b$-torsors $\mathcal{E}' = \mathcal{I}som_G(\mathcal{E}_b, \mathcal{E}_1)$ and $\mathcal{E}'_{\mathrm{triv}} = \mathcal{I}som_G(\mathcal{E}_b, \mathcal{E}_b)$ (the latter is the trivial $\mathcal{G}_b = \mathcal{I}som_G(\mathcal{E}_b, \mathcal{E}_b)$-torsor). Then, we take an inverse as above to reverse the arrows to get $\mathscr{M}_{\mathcal{E}'\to\mathcal{E}'_{\mathrm{triv}}} \xrightarrow{\sim} \mathscr{M}_{\mathcal{E}_{\mathrm{triv}}\to\mathcal{E}'}$. Composing the two isomorphisms yields

$$\mathscr{M}_{1\to b} \xrightarrow{\sim} \mathscr{M}_{\mathcal{E}'_{\mathrm{triv}}\to\mathcal{E}'}.$$

When b is basic so that $\mathcal{G}_b = G_b \times_E \mathcal{X}$, the right-hand side is canonically identified with $\mathscr{M}_{1\to b^{-1}}$ where here b^{-1} is viewed as an element of $G_b(\breve{E}) = G(\breve{E})$ (see, e.g., [13, §8.5]). This construction thus extends the classical two towers duality to the non-basic case (as well as the inscribed setting). Outside of the basic case, however, $\mathcal{G}_b$ is not the base change of a group over $\mathrm{Spec}\, E$, so one needs to allow the more general moduli spaces for $\mathcal{X}$-pure inner forms considered in this subsection in order to state it.

10. Cohomological smoothness

Let $E/\mathbb{Q}_p$ be a finite extension with residue field $\mathbb{F}_q$. In this section, we work in the inscribed context $(\mathrm{Spd}\mathbb{F}_q, X_{E,\square})$. The main purpose of this section is to prove Theorem E and Corollary F.

10.1. **Setup.** Let G/E be a connected reductive group, let $b \in G(\breve{E})$, and let $[\mu]$ be a conjugacy class of cocharacters of $G_{\overline{E}}$. As in Section 2.6, we write $\mathscr{M}_{b,[\mu]} \coloneqq \mathscr{M}_{1\to b,[\mu]}$ where the latter is as defined in Section 9.

We write G^{der} for the derived subgroup of G and $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$ for its abelianization. We write det for the projection $G \to G^{\mathrm{ab}}$. The push-out along det induces a map $D : \mathscr{M}_{b,[\mu]} \to \mathscr{M}_{\det(b),[\det(\mu)]}$.

Let $\mathbb{C}_p \coloneqq \overline{E}^{\wedge}$, and assume that $\mathscr{M}_{b,[\mu]}(\mathbb{C}_p) \neq \varnothing$, or equivalently that $b \in B(G,[\mu^{-1}])$. Then we can and do fix a $\mathbb{C}_p$-point $\tau : \mathrm{Spd}\mathbb{C}_p \to \mathscr{M}_{\det(b),[\det(\mu)]}$, and we let

$$\mathscr{M}^{\tau}_{b,[\mu]} \coloneqq D \times_{\mathscr{M}_{\det(b)},[\det(\mu)]} \tau.$$

It is an inscribed v-sheaf as a fiber product of inscribed v-sheaves, and admits a natural inclusion $\mathscr{M}^{\tau}_{b,[\mu]} \hookrightarrow \mathscr{M}_{b,[\mu]} \times_{\mathrm{Spd}\breve{E}([\mu])} \mathrm{Spd}\mathbb{C}_p$.

10.2. **The tangent bundle of $\mathscr{M}^{\tau}_{b,[\mu]}$.** Let $\mathrm{Lie}\, G = \mathrm{Lie}\, G^{\mathrm{der}} \oplus \mathrm{Lie}\, Z(G)$, and the kernel of the map $G \to G^{\mathrm{ab}}$ is $\mathrm{Lie}\, G^{\mathrm{der}}$. Writing $\mathfrak{g}^\circ = \mathrm{Lie}\, G^{\mathrm{der}}(E)$ and $\mathfrak{z}^\circ = \mathrm{Lie}\, Z(G)(E)$, we have $\mathfrak{g} = \mathfrak{g}^\circ \oplus \mathfrak{z}$ as a representation of G. Thus we may form the lattice $\mathfrak{g}^{\circ,+}_{\max} = \mathfrak{g}^\circ_b \otimes_{\breve{E}} \mathbb{B}^+_{\mathrm{dR}} + \mathfrak{g}^\circ \otimes_E \mathbb{B}^+_{\mathrm{dR}}$, and the associated common modification $\mathcal{E}^\circ_{\max}$ of $\mathfrak{g} \otimes_E \mathcal{O}_{\mathcal{X}}$ and $\mathcal{E}(\mathfrak{g}_b)$.

Lemma 10.2.1. *The isomorphism $T_{\mathscr{M}_{b,[\mu]}} = \mathrm{BC}(\mathcal{E}_{\max})$ of Theorem C restricts to an isomorphism $T_{\mathscr{M}^{\tau}_{b,[\mu]}} = \mathrm{BC}(\mathcal{E}^\circ_{\max})$*

Proof. Let D denote the map $\mathscr{M}_{b,[\mu]} \to M_{\det(b),[\det(\mu)]}$. By construction, $T_{\mathscr{M}^{\tau}_{b,[\mu]}}$ is the restriction to $\mathscr{M}^{\tau}_{b,[\mu]}$ of the kernel of

$$dD : T_{\mathscr{M}_{b,[\mu]}} \to \det{}^* T_{\mathscr{M}_{\det(b),[\det(\mu)]}}.$$

From the decomposition of $\mathfrak{g}$, it follows that $\ker dD$ is exactly $\mathrm{BC}(\mathcal{E}^\circ_{\max})$. □

Lemma 10.2.2. *For any algebraically closed perfectoid $C/\mathbb{F}_q$ and $z : \mathrm{Spd}C \to \mathscr{M}^{\tau}_{b,[\mu]}$, the induced bundle $\mathcal{E}^\circ_{\max,z}$ on $X_{E,C}$ has non-negative Harder-Narasimhan slopes.*

Proof. Since $\mathfrak{g}^\circ \otimes_E \mathcal{O}_{X_{E,C}^{\mathrm{alg}}} \subseteq \mathcal{E}^\circ_{\max,z}$ with torsion quotient, its Harder-Narasimhan slopes are non-negative — otherwise it would admit a nonzero morphism to a simple bundle $\mathcal{O}(\lambda)$ for $\lambda < 0$, but this would restrict to a non-zero morphism from the trivial bundle, which does not exist. □

It will be the dual bundle that is naturally related to the geometry of fibers of $\pi_1 \times \pi_2$. We write $\mathfrak{g}^{\circ,+}_{\min} := \mathfrak{g}^\circ_b \otimes_{\breve{E}} \mathbb{B}^+_{\mathrm{dR}} \cap \mathfrak{g}^\circ \otimes_E \mathbb{B}^+_{\mathrm{dR}}$, and let $\mathcal{E}^\circ_{\min}$ be the associated modification, i.e. the largest common modification contained in both $\mathfrak{g}^\circ \otimes_E \mathcal{O}_{\mathcal{X}}$ and $\mathcal{E}(\mathfrak{g}^\circ_b)$.

Lemma 10.2.3. $(\mathcal{E}^\circ_{\max})^* \cong \mathcal{E}^\circ_{\min}$.

Proof. We have

$$(\mathfrak{g}^{\circ,+}_{\max})^* = (\mathfrak{g}^\circ_b \otimes_{\breve{E}} \mathbb{B}^+_{\mathrm{dR}} + \mathfrak{g}^\circ \otimes_E \mathbb{B}^+_{\mathrm{dR}})^* = (\mathfrak{g}^\circ_b)^* \otimes_{\breve{E}} \mathbb{B}^+_{\mathrm{dR}} \cap (\mathfrak{g}^\circ)^* \otimes_E \mathbb{B}^+_{\mathrm{dR}}.$$

It follows that $(\mathcal{E}^\circ_{\max})^*$ is the modification of $(\mathfrak{g}^\circ)^* \otimes_E \mathcal{O}_{\mathcal{X}}$ associated to the lattice $(\mathfrak{g}^\circ_b)^* \otimes_{\breve{E}} \mathbb{B}^+_{\mathrm{dR}} \cap (\mathfrak{g}^\circ)^* \otimes_E \mathbb{B}^+_{\mathrm{dR}}$. Since $\mathfrak{g}^\circ$ is a semisimple lie algebra, the Killing form gives an isomorphism $\mathfrak{g}^\circ \cong (\mathfrak{g}^\circ)^*$ as a representation of G. Thus we obtain an isomorphism of this modification with the modification of $\mathfrak{g}^\circ \otimes_E \mathcal{O}_{\mathcal{X}}$ associated to the lattice $\mathfrak{g}^\circ_b \otimes_{\breve{E}} \mathbb{B}^+_{\mathrm{dR}} \cap \mathfrak{g}^\circ \otimes_E \mathbb{B}^+_{\mathrm{dR}} = \mathfrak{g}^{\circ,+}_{\min}$, i.e. $\mathcal{E}^\circ_{\min}$. □

Remark 10.2.4. Lemma 10.2.3 will not typically hold if G is not reductive. This is why we restrict to the reductive case in this section.

10.3. **Fibers of $\pi_1 \times \pi_2$.** Let $C/\mathbb{F}_q$ be an algebraically closed perfectoid and let $z : \mathrm{Spd}(C, \mathcal{O}_C) \to \mathscr{M}^\tau_{b,[\mu]}$. We write π_i^τ for the restriction of π_i to $\mathcal{M}^\tau_{b,[\mu]}$.

Lemma 10.3.1. *π_1 is a $G^{\mathrm{der}}(E^{\diamond\mathrm{lf}})$-torsor.*

Proof. This follows since the fibers of the associated period map for $\mathcal{M}_{\det(b),\det([\mu])}$ are $G^{\mathrm{ab}}(E^{\diamond\mathrm{lf}})$-torsors, so that the fibers of π_1^τ are torsors for the kernel $G^{\mathrm{der}}(E^{\diamond\mathrm{lf}})$ of $G(E^{\diamond\mathrm{lf}}) \to G^{\mathrm{ab}}(E^{\diamond\mathrm{lf}})$. □

We write $\pi^\tau := \pi_1^\tau \times \pi_2^\tau$ and

$$F_z := \pi^\tau \times_{\mathrm{Gr}_{[\mu]} \times \mathrm{Gr}_{[\mu^{-1}]}} \pi^\tau(z),$$

i.e. F_z is the intersection of the fibers of π_1^τ and π_2^τ that contain z, equipped with its natural inscribed structure.

Because $\mathcal{E}^\circ_{\min} \subseteq \mathcal{E}^\circ_{\max}$, there is a natural inclusion $\mathrm{BC}(\mathcal{E}^\circ_{\min}) \hookrightarrow \mathrm{BC}(\mathcal{E}^\circ_{\max})$.

Lemma 10.3.2. *The isomorphism $T_{\mathscr{M}^\tau_{b,[\mu]}} = \mathrm{BC}(\mathcal{E}^\circ_{\max})$ restricts to an isomorphism $T_{F_z} = \mathrm{BC}(\mathcal{E}^\circ_{\min})$.*

Proof. The tangent bundle T_{F_z} is the restriction to F_z of the kernel of $d\pi^\tau = d\pi_1^\tau \times d\pi_2^\tau$. From the computation of these derivatives in Theorem C, this is exactly $\mathrm{BC}(\mathcal{E}^\circ_{\min})$. □

We note that $F_z \hookrightarrow \pi_1^\tau \times_{\mathrm{Gr}_{[\mu]}} \pi_1^\tau(z)$. Since π_1^τ is a $G^{\mathrm{der}}(E^{\diamond\mathrm{lf}})$-torsor, we obtain an embedding over $\mathrm{Spd}C$, $F_z \hookrightarrow G^{\mathrm{der}}(E^{\diamond\mathrm{lf}}) \cdot z$. In fact we can be more precise:

Lemma 10.3.3. *For* $H_z := (\mathrm{Stab}_{G^{\mathrm{der}}}(\pi_2^\tau(z)))^{\diamond\mathrm{lf}} \times_{G^{\diamond\mathrm{lf}}} G(E^{\diamond\mathrm{lf}})$*,* $F_z = H_z \cdot z$*.*

Proof. The induced map $(\pi_1^\tau)^{-1}(z) = G^{\mathrm{der}}(E^{\diamond\mathrm{lf}}) \cdot z \xrightarrow{\pi_2^\tau} \mathrm{Gr}_{[\mu^{-1}]}$ sends $g \cdot z$ to $g \cdot \pi_2(z)$ (by the equivariance of π_2), so the claim is immediate from the definitions. □

Lemma 10.3.4. $H_z(\mathrm{Spd}C) \leq G(\mathbb{Q}_p)$ *is discrete if and only if* $(\mathrm{Lie}\, H_z)_0 = 0$*.*

Proof. By construction,

$$H := H_z(\mathrm{Spd}C) = \mathrm{Stab}_{G^{\mathrm{der}}(\mathbb{Q}_p)}(\pi_2^\tau(z))$$

Since H is a closed subgroup, it is a p-adic Lie group and thus discrete if and only if $\mathrm{Lie}\, H = 0$. By exponentiating, we find

$$\mathrm{Lie}\, H = \mathrm{Lie}\,\mathrm{Stab}_{G^{\mathrm{der}}(C^\sharp)}(\pi_2^\tau(z)) \cap \mathfrak{g}^\circ \subseteq \mathfrak{g}^\circ_{C^\sharp}.$$

But, by construction, this is exactly the underlying set of $(\mathrm{Lie}\, H_z)_0 \subseteq \underline{\mathfrak{g}}$. □

Proposition 10.3.5. *For* $z : \mathrm{Spd}(C) \to \mathscr{M}^\tau_{b,[\mu]}$ *as above, we write* $\mathcal{E}^\circ_{\max,z}$ *for the induced bundles on* $X^{\mathrm{alg}}_{E,C^\flat}$*. The Harder-Narasimhan slopes of* $\mathcal{E}^\circ_{\max,z}$ *are all non-negative, and the following are equivalent:*

1. $\mathcal{E}^\circ_{\max,z}$ *does not admit zero as a Harder-Narasimhan slope.*
2. $T_{F_z,z}(\mathrm{Spd}C) = 0$*.*
3. $(\mathrm{Lie}\, H_z)(\mathrm{Spd}C) = 0$
4. $\mathrm{Stab}_{G(\mathbb{Q}_p)}(\pi_2(z))$ *is a discrete subgroup of* $G(\mathbb{Q}_p)$*.*
5. $(F_z)(\mathrm{Spd}C)$ *is discrete subset of* $G(\mathbb{Q}_p) \cdot z$*.*

Proof. The non-negativity of slopes is Lemma 10.2.2.

The bundle $(\mathcal{E}^\circ_{\max,z})$ admits zero as a Harder-Narasimhan slope if and only if its dual does, and this dual is equal to $\mathcal{E}^\circ_{\min,z}$ by Lemma 10.3.2. As the dual of a bundle with non-negative slopes, its slopes are non-positive, thus it admits zero as a slope if and only if it has a non-zero global section. Invoking Lemma 10.3.2, this gives the equivalence between (1) and (2).

By Lemma 10.3.3, we obtain $T_{F_z,z} = \mathrm{Lie}\, H_z$, which gives the equivalence between (2) and (3). The equivalence between (3) and (4) follows from Lemma 10.3.4. Finally, the equivalence between (4) and (5) is again a consequence of Lemma 10.3.3. □

10.4. EL moduli problems.

In this section $E = \mathbb{Q}_p$.

Following [14, §2.2] (with a slight modification — see) we fix EL data $\mathcal{D} = (B, V, H, [\mu])$: B is a semisimple $\mathbb{Q}_p$-algebra, V is a finite dimensional B-module, H is a p-divisible group up to isogeny over $\overline{\mathbb{F}}_p$ equipped with an action $B \hookrightarrow \mathrm{End}(H)$, and $[\mu]$ is a conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$ for $G := \mathrm{GL}_B(V)$ such that

1. The rational covariant Dieudonné module $M(H)$, as a $B \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$-module, is isomorphic to $V \otimes \breve{\mathbb{Q}}_p$

(2) The action on $V \otimes \overline{\mathbb{Q}}_p$ of $\mathbb{G}_m$ induced by any $\mu \in [\mu]$ is by weights 0 and 1, and the weight 1 space has dimension equal to that of H.

Remark 10.4.1. Section 10.4 In [14] they also take the cocharacter to have weights 0 and 1, but require the weight 0 subspace have dimension equal to that of H. We have chosen our μ as the homological Hodge cocharacter, i.e. to lie in the conjugacy class that would act as multiplication by z on the Lie algebra of the associated lift of H (note that for the Hodge filtration on $M(H)$ associated to a lift, $\mathrm{gr}^{-1}M(H)$ is the Lie algebra; the Hodge filtration is thus the filtration attached to μ^{-1}).

We fix an isomorphism as in (1), so that the Frobenius on $M(H) = V \otimes \breve{\mathbb{Q}}_p$ is of the form $b\sigma$ for $b \in \mathrm{GL}_B(V)$. Thus we can consider the moduli of modifications $\mathscr{M}_{b,[\mu]}$ over $\breve{\mathbb{Q}}_p([\mu])$ as in Section 9.

We note that there is a $\mu \in [\mu]$ defined over $\breve{\mathbb{Q}}_p([\mu])$ — we fix such a μ, and write $V[-1] \subseteq V \otimes \breve{\mathbb{Q}}_p([\mu])$ for the associated -1 weight space.

Lemma 10.4.2. *$\mathcal{M}_{b,[\mu]}$ is isomorphic over $\mathrm{Spd}\breve{\mathbb{Q}}_p([\mu])$ to the functor sending $\mathcal{X}/X_P, P/\mathrm{Spd}\breve{\mathbb{Q}}_p([\mu])$ to the set of exact sequences of $B \otimes \mathcal{O}_{\mathcal{X}}$-modules*

$$0 \to V \otimes \mathcal{O}_{\mathcal{X}} \to \mathcal{E}_b(V) \to \infty_* W \to 0$$

where W is a $\mathcal{O}_{\mathcal{X}_{P^\sharp}} \otimes_{\mathbb{Q}_p} B$-module locally isomorphic to $V[-1] \otimes_{\breve{\mathbb{Q}}_p([\mu])} \mathcal{O}_{\mathcal{X}_{P^\sharp}}$.

Proof. Given a modification $\mathcal{E}_1 \to \mathcal{E}_b$ of type μ, we obtain by push-out along the representation V such an exact sequence. The inverse is given by sending such an exact sequence to the induced modification of G-torsors

$$\mathcal{E}_1 = \mathcal{I}som_{B\otimes\mathcal{O}_{\mathcal{X}}}(V \otimes \mathcal{O}_{\mathcal{X}}, V \otimes \mathcal{O}_{\mathcal{X}}) \dashrightarrow \mathcal{I}som_{B\otimes\mathcal{O}_{\mathcal{X}}}(V \otimes \mathcal{O}_{\mathcal{X}}, \mathcal{E}_b(V)) = \mathcal{E}_b.$$

□

Note that the underlying v-sheaf of the functor in Lemma 10.4.2 is precisely that functor appearing in [14, Proposition 3.2.3]. In [14, §5.6], Ivanov and Weinstein construct a smooth quasi-projective scheme $Z/X^{\mathrm{alg}}_{\mathbb{C}_p^\flat}$ such that $((Z/X^{\mathrm{alg}}_{\mathbb{C}_p^\flat})^{\diamond\mathrm{lf}})_0$ is isomorphic to the subfunctor corresponding to $(\mathscr{M}^{\tau}_{b,[\mu]})_0$. In fact, with no further work, we obtain:

Proposition 10.4.3. *For $Z/X_{\mathbb{C}_p^\flat}$ the smooth quasi-projective scheme constructed in [14, §5.6], $(Z/X^{\mathrm{alg}}_{\mathbb{C}_p^\flat})^{\diamond\mathrm{lf}} = (Z^{\mathrm{an}}/X_{\mathbb{C}_p^\flat})^{\diamond\mathrm{lf}} = \mathscr{M}^{\tau}_{b,[\mu]}$.*

Proof. That the smooth quasi-projective scheme and its analytification have the same moduli of sections is always true. The identification with $\mathscr{M}^{\tau}_{b,[\mu]}$ is immediate from the construction in [14, §5.6], which is by analytifyication of a smooth scheme that parameterizes such exact sequences with determinant τ for any scheme over $X^{\mathrm{alg}}_{\mathbb{C}_p^\flat}$ such that the pullback of ∞ is a Cartier divisor.

□

Consider a rank one geometric point $z : \mathrm{Spd}C \to \mathcal{M}_{b,[\mu]}$. Then, via the Scholze-Weinstein classification [24], $\pi_{\mathrm{HT}}(z)$ defines a p-divisible group up-to-isogeny G_z over $\mathcal{O}_{C^\sharp}$ with an inclusion $B \hookrightarrow \mathrm{End}(G_z)$. We write $A_z = \mathrm{End}_B(G)$. Note that, for F the center of B, we have $F \hookrightarrow A_z$. Following [14], we say z is non-special if $F \hookrightarrow A_z$ is an isomorphism.

Corollary 10.4.4. *There is a unique partially proper open locus* $(\mathscr{M}^\circ_{b,[\mu]})_0^{\mathrm{non-sp}} \subseteq (\mathscr{M}^\circ_{b,[\mu]})_0$ *whose rank one points are exactly the non-special points, and the map* $(\mathscr{M}^\circ_{b,[\mu]})_0^{\mathrm{non-sp}} \to \mathrm{Spd}\mathbb{C}_p$ *is cohomologically smooth.*

Proof. We first claim a point $z : \mathrm{Spd}C \to \mathscr{M}^\tau_{b,[\mu]}$ is non-special if and only if $\mathcal{E}^\circ_{\max,z}$ does not admit 0 as a Harder-Narasimhan slope. Indeed, by Proposition 10.3.5, the latter is equivalent to $(\mathrm{Lie}\, H_z)(\mathrm{Spd}C) = 0$. But, essentially by definition, nonzero elements of $\mathrm{Lie}\, H_z(\mathrm{Spd}C)$ correspond to noncentral elements in the B-linear endomorphisms of V preserving the Hodge-Tate filtration (using that $\mathrm{Lie}\,\mathrm{GL}_B(V) = \mathrm{End}_B(V) = F \oplus (\mathrm{Lie}\,\mathrm{GL}_b(V))^\circ$); by the Scholze-Weinstein classification, these exist if and only if z is special.

In particular, since the locus where $\mathcal{E}^\circ_{\max,z}$ does not admit 0 as a Harder-Narasimhan slope is partially proper and open (by the usual semi-continuity of the Newton polygon), we obtain the existences of $(\mathscr{M}^\circ_{b,[\mu]})_0^{\mathrm{non-sp}}$.

We now claim that $(\mathscr{M}^\circ_{b,[\mu]})_0^{\mathrm{non-sp}}$ is the locus $\mathcal{M}_Z^{\mathrm{sm}}$ appearing in [7, Definition IV.4.1]. Given this claim, the cohomological smoothness follows from the Jacobian criterion [7, Theorem IV.4.2]. To see this claim it suffices to check at rank one points z as above. For these note that, for s the associated section of Z^{an}, Example 5.4.3-(2) implies $\mathrm{BC}(s^*T_{Z^{\mathrm{an}}/X^\flat_{\mathbb{C}_p}}) = T_{(Z/E)^{\diamond \mathrm{lf}},z}$. But, by Proposition 10.4.3 and Lemma 10.2.1, this is also equal to $\mathrm{BC}(\mathcal{E}^\circ_{\max,z})$. Since $\mathrm{BC}(s^*T_{Z^{\mathrm{an}}/X^\flat_{\mathbb{C}_p}}) = \mathrm{BC}((s^*T_{Z^{\mathrm{an}}/X^\flat_{\mathbb{C}_p}})_{\geq 0})$, where the subscript denotes the non-negative slope part, by full faithfullness of the functor from vector bundles of non-negative slope to Banach-Colmez spaces [17], we deduce $(s^*T_{Z^{\mathrm{an}}/X^\flat_{\mathbb{C}_p}})_{\geq 0} = \mathcal{E}^\circ_{\max,z}$. But it is straightforward to check that both $\mathcal{E}^\circ_{\max,z}$ and $s^*T_{Z^{\mathrm{an}}/X^\flat_{\mathbb{C}_p}}$ have dimension equal to the dimension of G^{der}, thus we conclude $\mathcal{E}^\circ_{\max,z} = s^*T_{Z^{\mathrm{an}}/X^\flat_{\mathbb{C}_p}}$. The above discussion then shows that the space $\mathcal{M}_Z^{\mathrm{sm}}$ where $s^*T_{Z^{\mathrm{an}}/X^\flat_{\mathbb{C}_p}} = \mathcal{E}^\circ_{\max,z}$ has strictly positive slopes is equal to the space $(\mathscr{M}^\circ_{b,[\mu]})_0^{\mathrm{non-sp}}$, as desired. □

Remark 10.4.5. This notion of special is different than the notion of special implicit in [12, 13], which would say the Tannakian structure group of the associated admissible pair is smaller than the generic Tannakian structure group. When b is basic, the latter is a weaker notion (i.e. special in the sense of [14] implies special in the sense of [12, 13], but not vice versa); this is the usual point that the Tannakian structure group of an object can shrink without the endomorphism algebra changing. When b is not basic, the two notions are not directly comparable; to see that the previous implication no longer holds consider the example of the infinite level Serre-Tate moduli space corresponding to an ordinary elliptic curve. In this example, all points are special in the sense used in Corollary 10.4.4, but only the points corresponding to CM elliptic curves are special in the sense of [12, 13]. The issue here is that, if b is not basic, then endomorphisms of the p-divisible group do not always come from endomorphisms of the admissible pair.

11. Inscribed global Shimura varieties

In this section we construct the inscription on any global Shimura variety satisfying Conjecture 2.4.1, and prove Theorem D. We work in the inscribed context $(\mathrm{AffPerf}/\mathrm{Spd}\mathbb{F}_p, X_{\mathbb{Q}_p,\square})$ of Section 5.

We recall the notation of Section 2.4. Let (G,X) be a Shimura datum, and assume the maximal $\mathbb{R}$-split $\mathbb{Q}$-anisotropic central torus is equal to G. We fix a p-adic field L and an embedding $\mathbb{Q}([\mu]) \to L$, where $\mathbb{Q}([\mu])$ is the reflex field for (G,X), i.e. the field of definition of the conjugacy class of Hodge cocharacters $[\mu]$ (a subfield of $\mathbb{C}$ finite over $\mathbb{Q}$). For $K \le G(\mathbb{A}^\infty)$ a sufficiently small compact open, we write Sh_K for the associated Shimura variety of level K as a (smooth) rigid analytic variety over L, and Sh^*_K for its minimal compactification. For $K^p \le G(\mathbb{A}^{\infty p})$ compact open, we write

$$\mathrm{Sh}^\diamond_{K^p} := \varprojlim_{K_p} \mathrm{Sh}^\diamond_{K_pK^p}, (\mathrm{Sh}^*_{K^p})^\diamond := \varprojlim_{K_p} (\mathrm{Sh}^*_{K_pK^p})^\diamond$$

where the limits are over compact open subgroups $K_p \le G(\mathbb{Q}_p)$ such that K_pK^p is sufficiently small. The natural map $\mathrm{Sh}^\diamond_{K^p} \to (\mathrm{Sh}^*_{K^p})^\diamond$ is an open subdiamond.

We assume Conjecture 2.4.1 holds for our choice of (G,X) and L; in particular, for any compact open subgroup $K^p \le G(\mathbb{A}^{\infty p})$ we have an associated Igusa v-stack $\mathrm{Igs}^*_{K^p}$ on AffPerf with a map $\overline{\pi}_{\mathrm{HT}} : \mathrm{Igs}^*_{K^p} \to \mathrm{Bun}_G$.

11.1. Construction of $\mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p}$.

We now make precise the definition of $\mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p}$ described in Section 2.4.

We write $\mathrm{BL}^{\diamond\mathrm{lf}} : \mathrm{Gr}_G \to \mathcal{X}^*\mathrm{B}G$ for the modification map $m_{\mathcal{E}_{\mathrm{triv}},\mathrm{Id}}$ in the notation of Eq. (8.3.2.1). Its restriction to $\mathrm{Gr}_{[\mu^{-1}]}$ can be viewed as a map $\mathrm{Fl}_{[\mu]} = \mathrm{Gr}_{[\mu^{-1}]} \to \mathcal{X}^*\mathrm{B}G$, since the Bialynicki-Birula map $\mathrm{BB} : \mathrm{Gr}_{[\mu^{-1}]} \to \mathrm{Fl}^{\diamond\mathrm{lf}}_{[\mu]}$ is an isomorphism for $[\mu]$ minuscule. The induced map on underlying v-sheaves is the map BL appearing in Conjecture 2.4.1.

We write $\overline{\pi}^{\diamond\mathrm{lf}}_{\mathrm{HT}}$ for the composition of $\overline{\pi}_{\mathrm{HT}}$ (viewed as a map of trivially inscribed v-stacks) with the natural map $\mathrm{Bun}G \to \mathcal{X}^*\mathrm{B}G$ (by pullback along $\mathcal{X}/X_P$). We let

$$(\mathrm{Sh}^*_{K^p})^{\diamond\mathrm{lf}} := (\mathrm{Igs}^*_{K^p} \xrightarrow{\overline{\pi}^{\diamond\mathrm{lf}}_{\mathrm{HT}}} \mathcal{X}^*\mathrm{B}G) \times_{\mathcal{X}^*\mathrm{B}G} (\mathrm{Fl}_{[\mu]} \xrightarrow{\mathrm{BL}^{\diamond\mathrm{lf}}} \mathcal{X}^*\mathrm{B}G). \quad (11.1.0.1)$$

Since we have assumed Conjecture 2.4.1, we have an identification

$$((\mathrm{Sh}^*_{K^p})^{\diamond\mathrm{lf}})_0 = (\mathrm{Igs}^*_{K^p} \xrightarrow{\overline{\pi}^\diamond_{\mathrm{HT}}} \mathrm{Bun}G) \times_{\mathrm{Bun}G} (\mathrm{Fl}_{[\mu]} \xrightarrow{\mathrm{BL}} \mathrm{Bun}G) = (\mathrm{Sh}^*_{K^p})^\diamond.$$

We may thus define $\mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}}$ as the open inscribed subdiamond

$$\mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}} := (\mathrm{Sh}_{K^p}^*)^{\diamond\mathrm{lf}} \times_{(\mathrm{Sh}_{K^p}^*)^\diamond} \mathrm{Sh}_{K^p}^\diamond$$

(where $(\mathrm{Sh}_{K^p}^*)^\diamond$ and $\mathrm{Sh}_{K^p}^\diamond$ are equipped with the trivial inscription).

By construction, there is a map $\pi_{\mathrm{HT}}^{\diamond\mathrm{lf}} : (\mathrm{Sh}_{K_p}^*)^{\diamond\mathrm{lf}} \to \mathrm{Fl}_{[\mu]}^{\diamond\mathrm{lf}}$ given by projection to the second factor and whose map on underlying v-sheaves is $\pi_{\mathrm{HT}}^\diamond$.

Lemma 11.1.1. *The action of $G(\mathbb{Q}_p^{\diamond\mathrm{lf}})$ on $\mathrm{Fl}_\mu^{\diamond\mathrm{lf}}$ preserves* BL, *thus induces*

$$a : (\mathrm{Sh}_{K^p}^*)^{\diamond\mathrm{lf}} \times G(\mathbb{Q}_p^{\diamond\mathrm{lf}}) \to \mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}}$$

such that π_{HT} is $G(\mathbb{Q}_p^{\diamond\mathrm{lf}})$-equivariant. It restricts to an action on $\mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}}$.

Proof. The existence of the action on $(\mathrm{Sh}_{K^p}^*)^{\diamond\mathrm{lf}}$ follows from Lemma 8.3.3, since $G(\mathbb{Q}_p^{\diamond\mathrm{lf}}) \subseteq G(\mathbb{B}_e)$ preserves $\mathrm{Fl}_\mu^{\diamond\mathrm{lf}} = \mathrm{Gr}_{[\mu^{-1}]} \subseteq \mathrm{Gr}_G$. It restricts to an action on $\mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}}$ because $\mathrm{Sh}_{K^p}^\diamond$ is preserved by the action of $G(\underline{\mathbb{Q}_p}) = G(\mathbb{Q}_p^{\diamond\mathrm{lf}})_0$. □

11.2. **Computation of the tangent bundle.** As in the case of local Shimura varieties and more general moduli of modifications treated in Section 9, the computation is accomplished by first treating an unbounded analog, and then cutting out the result we are interested in within.

To that end, let $\mathcal{S} : (\mathrm{Igs}_{K^p}^* \xrightarrow{\overline{\pi}_{\mathrm{HT}}^{\diamond\mathrm{lf}}} \mathcal{X}^*\mathrm{B}G) \times_{\mathcal{X}^*\mathrm{B}G} (\mathrm{Gr}_G \xrightarrow{\mathrm{BL}^{\diamond\mathrm{lf}}} \mathcal{X}^*\mathrm{B}G)$.

Lemma 11.2.1. *$\mathcal{S}$ is an inscribed v-sheaf.*

Proof. Since each of the terms in the fiber product is an inscribed v-stack, it is an inscribed v-stack by Lemma 4.1.7. It is a sheaf rather than a more general stack because Gr_G is a sheaf and $\overline{\pi}_{\mathrm{HT}}^{\diamond\mathrm{lf}}$ is 0-truncated. □

We note that on $\mathcal{S}$ we have the trivial bundle $\mathfrak{g} \otimes \mathcal{O}_{\mathcal{X}}$ over $\mathcal{X}$ and the pullback of the universal bundle associated to the adjoint representation over $\mathcal{X}^*\mathrm{B}G$, $\mathfrak{g}_{\mathrm{univ}}$. By the construction of the map $\mathrm{BL}^{\diamond\mathrm{lf}}$, we have fixed an isomorphism between these two bundles after restriction to $\mathcal{X}\backslash\infty$.

By Lemma 8.3.3, $\mathcal{S}$ is a $G(\mathbb{B}_e)$ quasi-torsor over $\mathrm{Igs}_{K^p}^*$ (in fact a torsor since the map to $\mathrm{Igs}_{K^p}^*$ is surjective, but we will not actually need this fact). Since $\mathrm{Igs}_{K^p}^*$ is trivially inscribed, we deduce that, writing $a : \mathcal{S} \times G(\mathbb{B}_e) \to \mathcal{S}$ for the action map, da_e induces an isomorphism $\mathfrak{g} \otimes \mathbb{B}_e = \mathrm{Lie}\, G(\mathbb{B}_e) \xrightarrow{\sim} T_{\mathcal{S}}$.

The map $\pi_2 : \mathcal{S} \to \mathrm{Gr}_G$ is equivariant along the inclusion $G(\mathbb{B}_e) \subseteq G(\mathbb{B}_{\mathrm{dR}})$. We note also that, by construction, $\pi_2^* \mathfrak{g}_{\mathrm{univ}}^+ = \mathfrak{g}_{\mathrm{univ}} \boxtimes \mathbb{B}_{\mathrm{dR}}^+$. Comparing with Corollary 7.1.6, we find $d\pi_2$ can be identified with the natural map $\mathfrak{g} \otimes \mathbb{B}_e \to \mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}} / \mathfrak{g}_{\mathrm{univ}} \boxtimes \mathbb{B}_{\mathrm{dR}}^+$. Since

$$(\mathrm{Sh}_{K^p}^*)^{\diamond\mathrm{lf}} = (\mathcal{S} \xrightarrow{\pi_2} \mathrm{Gr}_G) \times_{\mathrm{Gr}_G} \mathrm{Gr}_{[\mu^{-1}]},$$

comparing with Proposition 7.2.3 we obtain

$$T_{(\mathrm{Sh}_{K^p}^*)^{\diamond\mathrm{lf}}} = \mathfrak{g} \otimes \mathbb{B}_e \times_{\mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}} / \mathfrak{g}_{\mathrm{univ}}^+} \mathfrak{g}_{\max}^+ / \mathfrak{g}_{\mathrm{univ}}^+ = \mathrm{BC}(\mathcal{E}_{\max}),$$

where $\mathfrak{g}_{\max}^+ = \mathfrak{g} \otimes \mathbb{B}_{\mathrm{dR}}^+ + \mathfrak{g}_{\mathrm{univ}} \boxtimes \mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{E}_{\max}$ is the associated modification of $\mathfrak{g} \otimes \mathcal{O}_{\mathcal{X}}$. We have thus established

Lemma 11.2.2. *With notation above, the $\mathbb{B}_{\mathrm{dR}}^{+}$-module on $\mathrm{Sh}_{K_p}^{\diamond\mathrm{lf}}$*

$$\mathfrak{g}_{\max}^{+} := \mathfrak{g}_{\mathrm{univ}} \boxtimes \mathbb{B}_{\mathrm{dR}}^{+} + \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^{+} \subseteq \mathfrak{g}_{\mathrm{univ}} \boxtimes \mathbb{B}_{\mathrm{dR}} = \mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}$$

is locally free. In particular, there is a vector bundle $\mathcal{E}_{\max}$ on $\mathcal{X}$ over $\mathrm{Sh}_{K_p}^{\diamond\mathrm{lf}}$ fitting into two canonical modification exact sequences of sheaves on $\mathcal{X}$

$$0 \to \mathfrak{g} \otimes_E \mathcal{O}_{\mathcal{X}} \to \mathcal{E}_{\max} \to \infty_* \left(\mathfrak{g}_{\max}^{+}/\mathfrak{g} \otimes_E \mathbb{B}_{\mathrm{dR}}^{+}\right) \to 0 \tag{11.2.2.1}$$

$$\text{and } 0 \to \mathfrak{g}_{\mathrm{univ}} \to \mathcal{E}_{\max} \to \infty_* \left(\mathfrak{g}_{\max}^{+}/\mathfrak{g}_{\mathrm{univ}} \boxtimes \mathbb{B}_{\mathrm{dR}}^{+}\right) \to 0. \tag{11.2.2.2}$$

There is a canonical identification $\mathrm{BC}(\mathcal{E}_{\max}) = T_{\mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}}}$ such that the v-sheafification of the associated long exact sequences of cohomology for Eq. (2.4.4.2) is identified with

(11.2.2.3)

$$0 \to \mathrm{BC}(\mathfrak{g}_{\mathrm{univ}}) \to \mathrm{BC}(\mathfrak{g}_{\max}) = T_{(\mathrm{Sh}_{K^p}^{\diamond})^{\diamond\mathrm{lf}}} \xrightarrow{d\pi_{\mathrm{HT}}^{\diamond\mathrm{lf}}} (\pi_{\mathrm{HT}}^{\diamond\mathrm{lf}})^{*} T_{\mathrm{Fl}_{[\mu]}^{\diamond\mathrm{lf}}} \to \mathrm{BC}(\mathfrak{g}_{\mathrm{univ}}[1]) \to 0.$$

11.3. **Interlude: Flat sections and Hodge period maps.** Let $L/\mathbb{Q}_p$ be an arbitrary non-archimedean extension. For this subsection we work in the inscribed context $(\mathrm{Spd}L, \square^{\sharp})$.

Let Z/L be smooth rigid analytic variety. Let G/L be a linear algebraic group, and let $(\mathcal{G}, \nabla)/Z$ be a G-torsor with integrable connection, i.e. an exact tensor functor from $\mathrm{Rep}G$ to vector bundles with integrable connection on Z. We write $\mathcal{G}$ for the underlying G-torsor, and $\mathcal{G}^{\diamond\mathrm{lf}}$ for the inscribed v-sheaf attached to the associated geometric G-torsor.

We write $G^{\diamond}$ for G with its trivial inscription. We are going to construct a natural reduction of structure group on $\mathcal{G}^{\diamond\mathrm{lf}}$ from a $G^{\diamond\mathrm{lf}}$-torsor over $Z^{\diamond\mathrm{lf}}$ to a $G^{\diamond}$-torsor $(\mathcal{G}, \nabla)^{\diamond\mathrm{lf}} \subseteq \mathcal{G}^{\diamond\mathrm{lf}}$ of flat sections over $Z^{\diamond\mathrm{lf}}$.

To that end, we write $r : Z^{\diamond\mathrm{lf}} \to Z^{\diamond} = (Z_0)^{\mathrm{triv}}$ for the natural map as in Definition 4.1.5, which maps $f : \mathcal{P} \to Z \in Z^{\diamond\mathrm{lf}}(\mathcal{P}/P^{\sharp})$ to $f|_{P^{\sharp}}$. We claim that the integrable connection induces a $G^{\diamond}$-equivariant map $\exp_{\nabla} : r^{-1}\mathcal{G}^{\diamond} \to \mathcal{G}^{\diamond\mathrm{lf}}$. Indeed, to give a section of $r^{-1}\mathcal{G}^{\diamond}$ on $\mathcal{P}/P^{\sharp}$ is to give a map $f : \mathcal{P} \to Z$ over $\mathrm{Spa}L$ and a section s_0 of $\mathcal{G}$ over $f|_{P^{\sharp}}$, and the integrable connection promotes this uniquely (and $G^{\diamond}$-equivariantly) to a flat section of s of $\mathcal{G}$ over f. We define $(\mathcal{G}, \nabla)^{\diamond\mathrm{lf}}$ to be the image of $r^{-1}\mathcal{G}^{\diamond}$ under $\exp_{\nabla}$.

Remark 11.3.1. In other words, we have shown $\exp_{\nabla}$ induces

$$r^{-1}\mathcal{G}^{\diamond} \times^{G^{\diamond}} G^{\diamond\mathrm{lf}} = \mathcal{G}^{\diamond\mathrm{lf}}.$$

Suppose that $\mathcal{G}$ is furthermore equipped with a filtration, so that we obtain a period map $\mathcal{G} \to \mathrm{Fl}$. We write π_{Hdg} for the restriction of the period map to $(\mathcal{G}, \nabla)^{\diamond\mathrm{lf}}$, and we write $\kappa_{(\mathcal{G},\nabla,\mathrm{Fil})}$ for the associated Kodaira-Spencer map, a $\mathcal{O}_Z$-linear homomorphism

$$T_Z \xrightarrow{\kappa_{(\mathcal{G},\nabla,\mathrm{Fil})}} \mathcal{G}(\mathfrak{g})/\mathrm{Fil}^0(\mathcal{G}(\mathfrak{g})).$$

It can be defined as follows: étale locally, we may choose another connection ∇' on $\mathcal{G}$ that preserves the filtration. The difference $\nabla - \nabla'$ assigns to any tangent vector t and representation V a endomorphism $f_{t,V} = (\nabla_{t,V} - \nabla'_{t,V})$ of V, functorially in v and compatibly with the tensor product in that $f_{t,V_1\otimes V_2} = f_{t,V_1} \otimes 1 + 1 \otimes f_{t,V_2}$. By the Tannakian formalism, it is given by an element $f_t \in \mathcal{G}(\mathfrak{g})$ (which maps to $\mathcal{G}(\mathrm{End}(V)) = \mathcal{E}nd(\mathcal{G}(V))$ for any V), and the map $t \mapsto f_t$ is a homomorphism $T_Z \to \mathcal{G}(\mathfrak{g})$ on the étale cover where ∇' was chosen. The composition of $t \mapsto f_t$ with projection to $\mathcal{G}(\mathfrak{g})/\mathrm{Fil}^0(\mathcal{G}(\mathfrak{g}))$ does not depend on the choice of ∇', thus descends to give $\kappa_{(\mathcal{G},\nabla,\mathrm{Fil})}$.

Lemma 11.3.2. *Let $t : (\mathcal{G},\nabla)^{\diamond\mathrm{lf}} \to Z^{\diamond\mathrm{lf}}$ denote the structure map. Then $dt : T_{(\mathcal{G},\nabla)^{\diamond\mathrm{lf}}} \to t^*T_{Z^{\diamond\mathrm{lf}}}$ is an isomorphism, and $d\pi_{\mathrm{Hdg}}$ is the composition*

$$T^{\diamond\mathrm{lf}}_{(\mathcal{G},\nabla)} = t^*T_{Z^{\diamond\mathrm{lf}}} = t^*(T_Z)^{\diamond\mathrm{lf}} \xrightarrow{\kappa_{(\mathcal{G},\nabla,\mathrm{Fil}^\bullet)}} \mathfrak{g}\otimes\mathcal{O}/\mathrm{Fil}^0(\mathfrak{g}\otimes\mathcal{O}) = \pi^*_{\mathrm{Hdg}}T_{\mathrm{Fl}}$$

where on the right we use the canonical trivialization $\mathcal{G}(\mathfrak{g}) = \mathfrak{g}\otimes\mathcal{O}$ over $\mathcal{G}$.

Remark 11.3.3. If we extend our definition of relative tangent bundles to inscribed v-stacks, then we obtain an equivalence $T_{\mathrm{Fl}^{\diamond\mathrm{lf}}/G^\diamond} = T_{(\mathrm{Fl}^{\diamond\mathrm{lf}}/G^\diamond)/(\mathrm{Fl}^\diamond/G^\diamond)}$ as in Example 4.6.2. Since the fibers of $(\mathrm{Fl}^{\diamond\mathrm{lf}}/G^\diamond) \to (\mathrm{Fl}^\diamond/G^\diamond)$ are discrete, the latter has a natural structure of an $\mathcal{O}$-module over $\mathrm{Fl}^{\diamond\mathrm{lf}}/G^\diamond$. In particular, we find $d(\pi^{\diamond\mathrm{lf}}_{\mathrm{Hdg}}/G^\diamond) = \kappa_{(\mathcal{G},\nabla,\mathrm{Fil})}$.

Proposition 11.3.4. *If $\kappa_{(\mathcal{G},\nabla,\mathrm{Fil})}$ is locally a direct summand, then the following square is Cartesian*

$$\begin{array}{ccc} (\mathcal{G},\nabla)^{\diamond\mathrm{lf}} & \xrightarrow{\pi^{\diamond\mathrm{lf}}_{\mathrm{Hdg}}} & \mathrm{Fl}^{\diamond\mathrm{lf}} \\ \downarrow & & \downarrow \\ \mathcal{G}^\diamond & \xrightarrow[\pi^\diamond_{\mathrm{Hdg}}]{} & \mathrm{Fl}^\diamond \end{array}$$

Proof. We fix a map $P^\sharp \to \mathcal{G}$ above $f : P^\sharp \to Z$. We let $\Gamma^\wedge_f$ denote the formal neighborhood of the graph of f. We obtain an induced map $\Gamma^\wedge_f \to \Gamma^\wedge_{\pi_{\mathrm{Hdg}}\circ f}$. Its derivative is $f^*\kappa_{(\mathcal{G},\nabla,\mathrm{Fil})}$, thus it is injective by our assumption. This implies the result: indeed, to give a map $\mathcal{P}/P^\sharp \to (\mathcal{G},\nabla)^{\mathrm{lf}}$ is the same as to give a map $g : P^\sharp \to \mathcal{G}$ lying above $f : P^\sharp \to Z$ and a deformation of f to $\tilde{f} : \mathcal{P} \to Z$. The graph of any such deformation factors through $\Gamma^\wedge_f$, and thus by the above computation is uniquely determined by its image in $\Gamma^\wedge_{\pi_{\mathrm{Hdg}}\circ f}$, which is equivalent to the image of the point in $\mathrm{Fl}^{\diamond\mathrm{lf}}$. □

11.4. **The Hodge period map at infinite level.** We note that we can define a natural *Hodge* period map on $\mathrm{Sh}^{\diamond\mathrm{lf}}_{K^p}$. Indeed, already on $\mathrm{Fl}^{\diamond\mathrm{lf}}$, we can define let $\mathcal{E}_{\mathrm{dR}} \coloneqq \infty^*(\mathrm{BL}^*\mathcal{E}_{\mathrm{univ}})$, a $G(\mathcal{O})$-torsor equipped with a Hodge filtration $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\mathcal{E}_{\mathrm{dR}}$ of type μ^{-1} by the modification $\mathcal{E}_{\mathrm{triv}} \dashrightarrow \mathcal{E}_{\mathrm{dR}}$. Then, because $\pi_{\overline{\mathrm{HT}}}$ factors through $\mathrm{Bun}G$, the pullback of $\mathcal{E}_{\mathrm{dR}}$ is equipped with a canonical "connection" (cf. Remark 2.1.3)

$$\mathcal{E}_{\mathrm{dR},0} \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}$$

where here $\mathcal{E}_{\mathrm{dR},0}$ denotes the pullback of $\mathcal{E}_{\mathrm{dR}}$ along $\mathcal{S} \to \mathcal{S}_0 \to \mathcal{S}$. In particular, by taking the image of the underlying $G^\diamond$-torsor of "flat sections" for $\mathcal{E}_{\mathrm{dR},0}$, we obtain a reduction of structure group on $\mathcal{E}_{\mathrm{dR}}$ to $G^\diamond$-torsor $\mathcal{E}_{\mathrm{dR}}^{\mathrm{flat}}$ over $\mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}}$, and then a period map classifying the Hodge filtration

$$\pi_{\mathrm{Hdg}} : \mathcal{E}_{\mathrm{dR}}^{\mathrm{flat}} \to \mathrm{Fl}_{[\mu^{-1}]}.$$

It follows from the comparison between the standard de Rham local systems and those constructed via p-adic Hodge theory as established in [18] that, on underlying v-sheaves, π_{Hdg} restricts to the pullback to $\mathrm{Sh}_{K_p}^{\diamond}$ of the usual Hodge period map for the de Rham G-torsor $\mathcal{G}_{\mathrm{dR}}$-over Sh_{K_p}. The associated Kodaira-Spencer map is, by construction, an isomorphism, thus Proposition 11.3.4 applies. Thus, the period map π_{Hdg} on $\mathcal{E}_{\mathrm{dR}}^{\mathrm{flat}}$ combined with the $\pi_{K_p}^{\diamond}$ induces a $G^\diamond$-equivariant map

$$\mathcal{E}_{\mathrm{dR}}^{\mathrm{flat}} \to (\mathcal{G}_{\mathrm{dR}}, \nabla)^{\diamond\mathrm{lf}},$$

and quotienting by $G^\diamond$ gives us $\pi_{K_p}^{\diamond\mathrm{lf}}$. Unwinding the construction we find it is a quasi-torsor for $K_p^{\diamond\mathrm{lf}}$ and, by applying Proposition 6.4.3, in fact a torsor (i.e. surjective) after restriction to $X_{\mathbb{Q}_p,\square}^{\mathrm{lf}^+}$. Comparing to the computations of Lemma 11.2.2 and the usual p-adic comparisons, we obtain:

Lemma 11.4.1. *After restricting to $X_{\mathbb{Q}_p,\square}^{\mathrm{lf}^+}$, the map $\pi_{K_p}^{\diamond\mathrm{lf}} : \mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}} \to \mathrm{Sh}_{K_pK^p}^{\diamond\mathrm{lf}}$ is a $K_p^{\diamond\mathrm{lf}}$-torsor. Moreover, the v-sheafification of the long exact sequence associated to Eq.* (11.2.2.1) *is canonically identified with*

$$0 \to \mathfrak{g}^{\diamond\mathrm{lf}} \xrightarrow{da_e} T_{\mathrm{Sh}_{K^p}^{\diamond\mathrm{lf}}} \xrightarrow{d\pi_{K_p}^{\diamond\mathrm{lf}}} T_{\mathrm{Sh}_{K_pK^p}^{\diamond\mathrm{lf}}} \to 0.$$

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