Last time: We defined p-adic Hodge structure, explained how it relates to cohomology of varieties, connection w/ p-adic HT à la Fontaine.

Today: Geometrize p-adic HT ~ equivariant vector bundles on the Fargues-Fontaine curve.

$\mathbb{C}_p \rightsquigarrow \mathcal{O}_{\mathbb{C}_p}^\flat \ni \varpi = (p, p^{1/p}, p^{1/p^2}, p^{1/p^3}, \ldots)$.

$W(\mathcal{O}_{\mathbb{C}_p}^\flat) \subset W(\mathbb{C}_p^\flat)$

$\mathbb{C}_p^\flat = \mathrm{Frac}\, \mathcal{O}_{\mathbb{C}_p}^\flat \leftarrow$ has $|\ |$ with valuation ring $\mathcal{O}_{\mathbb{C}_p}^\flat$.

$|(a_0, a_1, \ldots)| = |a_0|_p$.

$A_{inf} := W(\mathcal{O}_{\mathbb{C}_p}^\flat)$.

Topology is I adic for $I = (p, [\varpi])$.

$\mathrm{Spa}(A_{inf}, A_{inf})$. $\leftarrow$ points correspond to continuous maps $A_{inf} \xrightarrow{x} K$

K is a field complete w.r.t. a valuation s.t. $|f(x)| \leq 1 \ \forall f \in A_{inf}$.

This lets you define sets like $\{x \mid |f(x)| \leq |g(x)|\}$ for $f, g \in A_{inf}$.

e.g. if we take $|f(x)| \leq 1$.

functions $A_{inf}\langle f \rangle$

$\sum a_i f^i$ $\quad a_i \to 0$.

Examples of points of $\mathrm{Spa}(A_{inf}, A_{inf})$.

$\mathcal{O}_{\mathbb{C}_p}^\flat \to \overline{\mathbb{F}}_p$ mod $\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}^\flat}$ $\longrightarrow W(\mathcal{O}_{\mathbb{C}_p}^\flat) \to W(\overline{\mathbb{F}}_p)$

$\to \overline{\mathbb{Q}}_p$ ⤳ $|\ |_p$

$\to \overline{\mathbb{F}}_p$ ⤳ $|\ |_{triv}$

$|p| \neq 0$
$|[\varpi]| = 0$.

$$W(\mathcal{O}_{\mathbb{C}_p^\flat}) \to \mathcal{O}_{\mathbb{C}_p^\flat} \to \mathbb{C}_p^\flat$$

$|p| = 0$

$|[\varpi]| \neq 0$

$\uparrow$ $|\ |$ from above.

throw out later.

$$W(\mathcal{O}_{\mathbb{C}_p^\flat}) \xrightarrow{\theta} \mathcal{O}_{\mathbb{C}_p} \to \mathbb{C}_p$$

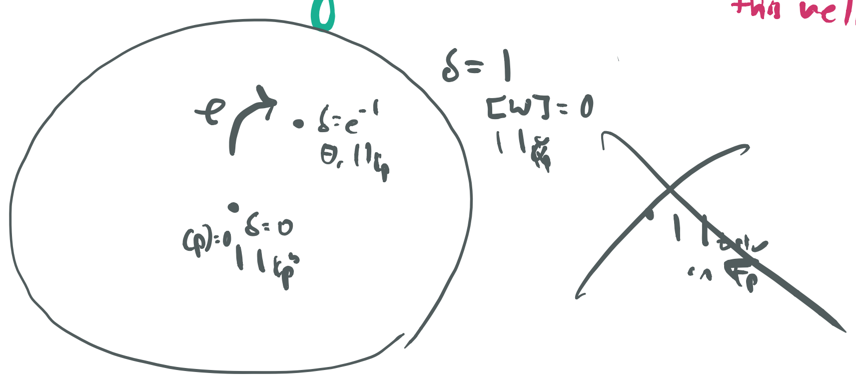
$|p| = |[\varpi]|$ $\qquad |\ |_p$

Think about absolute values on A_{inf} by comparing $|p|$ & $|[\varpi]|$

$$\delta(x) = -\frac{\log_p |p(\tilde{x})|}{\log_p |[\varpi](\tilde{x})|}$$

$\leftarrow$ some modification to get standard abs. value.

$\leftarrow$ quotient makes this well-defined



Frob $(x \mapsto x^p)$ on $\mathcal{O}_{\mathbb{C}_p^\flat}$ induces an aut. φ

$W(\mathcal{O}_{\mathbb{C}_p^\flat}) = A_{inf}$. $\qquad \mathcal{Y} = \mathrm{Spa}(A_{inf}, A_{inf}) \setminus$ triv. val. on $\mathbb{F}_p$.

$\varphi : Y \to Y$.

$\delta(\varphi(x)) = \delta(x)^{1/p}$.

Only fixed points are $\delta = 0$ and $\delta = 1$.

$\hookrightarrow \varphi$ acts prop. discont.

on $Y = \mathcal{Y} \setminus \delta=0 \ \ \delta=1$

$= \mathcal{Y} \setminus V(p[\varpi])$.

Fix $\overset{I}{\overset{\shortparallel}{[a,b]}} \subseteq [0,1]$. $\qquad a \neq 0 \qquad b \neq 1$.

What are functions on Y_I $\quad a \le \delta \le b$

$$A_{inf}\left[\frac{1}{p[\varpi]}\right]$$
$$\|$$
$$W(\mathbb{C}_p^b)^{bdd}\left[\frac{1}{p}\right]$$

$\uparrow$ meromorphic functions of p with bdd coefficients in $\mathbb{C}_p^b$.

Get $\mathcal{O}(Y_I)$ by completing for

$$|\ \ |_\rho \quad \rho = a, b.$$

$$\left|\sum_{n \gg 0} [a_n] p^n\right|_\rho = \sup_n |a_n| \rho^n.$$

$\uparrow$ maximum value of this function $\delta = \rho$.

$X = Y/e^{\mathbb{Z}}$ $\leftarrow$ Fargues Fontaine curve

$G_{\mathbb{Q}_p} \circlearrowright X$ (acts on Y, commuting w/ $e^{\mathbb{Z}}$).

$$H^0(X, \mathcal{O}) = \mathbb{Q}_p.$$

(Why?) $\leftarrow$ $= \mathcal{O}(Y_{(0,1)})^{e=1}$

Idea: $\left(W(\mathbb{C}_p^b)[\frac{1}{p}]\right)^{e=1} = \mathbb{Q}_p$.

(because $(\mathbb{C}_p^b)^{Frob=1}$ is $\mathbb{F}_p$).

If V is a representation of $G_{\mathbb{Q}_p}$ on a $\mathbb{Q}_p$-vector space.

Then $V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$ ← an eq. vector bundle.

Can show that this is fully faithful.

$$\mathrm{Rep}\, G_{\mathbb{Q}_p} \longleftrightarrow G_{\mathbb{Q}_p}\text{-eq. vector bundles on } X_{FF}.$$

All of Fontaine's comparison theorems are of the form

$$V \longrightarrow \left(V \otimes_{\mathbb{Q}_p} \mathcal{O}_X\right)\Big|_U$$

U ← locus on X or Y

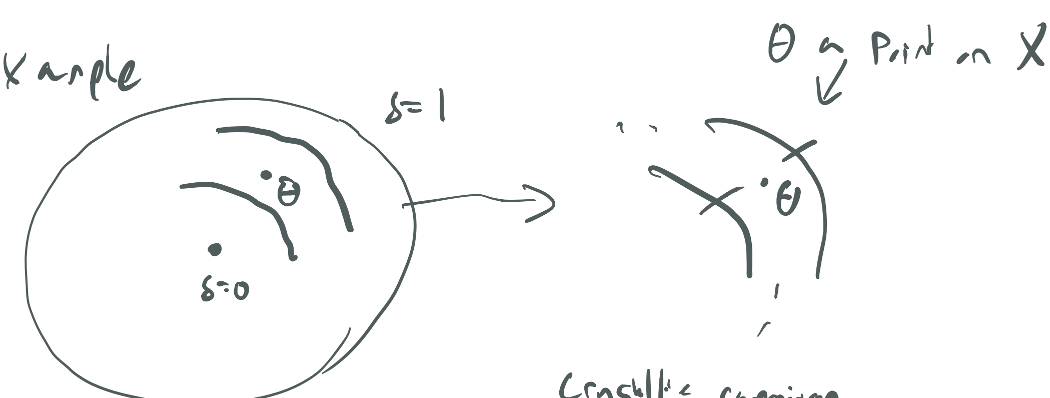
$\cong$

Simple equivariant vector bundle.

Vector bundles on X $\longleftrightarrow$ Isocrystals $= \breve{\mathbb{Q}}_p$-vector spaces with semilinear ϕ

$$(W, \varphi_W) \mapsto \left(W \otimes_{\breve{\mathbb{Q}}_p} \mathcal{O}_Y\right)^{(\varphi_W \otimes e) = 1}.$$

Example



Crystalline comparison

certain V

$(\Phi_w \otimes e) = 1$

$$V \otimes \mathcal{O}_X|_{X \setminus \{\theta\}} \cong (W \underset{\mathbb{Q}_p}{\otimes} \mathcal{O}_Y)|_{X \setminus \{\theta\}}$$

$G_{\mathbb{Q}_p}$-eq.

where W has the trivial Galois action.

This a $G_{\mathbb{Q}_p}$-eq. meromorphic modification at θ.

$\Updownarrow$

determined by a $\mathbb{B}_{dR}^+$-lattice in $W \otimes \mathbb{B}_{dR}$.

Completed local ring of X at θ is $\mathbb{B}_{dR}^+$.

$\Updownarrow$

determined by the trace filtration on W because Galois-equivariant.