$\underline{\text{Last time}}$ $\mathbb{B}_{dR}^+ \subseteq \mathbb{B}_{dR} = \mathbb{B}_{dR}^+[\frac{1}{t}]$ $t = \log[\varepsilon]$ $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ $\in \mathcal{O}^\flat$

$\uparrow$ Complete dvr w/ residue field $\mathbb{C}_p$ $\nwarrow$ Transforms via $G_{\mathbb{Q}_p}$ under cyclotomic character.

$\underline{\text{Def'n}}$: A $\breve{\mathbb{Q}}_p$ - p-adic Hodge structure $\leftarrow$ Not standard notation.
$\breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$
(like a $\mathbb{R}$-Hodge structure).

is a pair $(V, \mathcal{L})$ where V is a $\breve{\mathbb{Q}}_p$-vector space an $\mathcal{L}$ is a $\mathbb{B}_{dR}^+$ lattice in $V \otimes \mathbb{B}_{dR}$.

(Why?) - This the structure you naturally get on the cohomology of a smooth projective* rigid analytic space $/ \mathbb{C}_p$

* $\leftarrow$ Maybe necessary to impose a good reduction hypothesis.

Result of Bhatt-Morrow-Scholze
Bhatt-Scholze.
(Essentially one way to encode prismatic cohomology).

$X / \mathbb{C}_p$ smooth proj. variety (+ reduction hypothesis)

$V = H^i_{et}(X, \mathbb{Q}_p)$

$\uparrow$ like Tate module of elliptic curve.

$V \otimes \mathbb{B}_{dR} \supseteq \mathcal{L}$ = canonical deformation of

$H^i_{dR}(X) =: \mathbb{H}^i(\Omega^\bullet_{X/\mathbb{C}_p})$.

$\uparrow$ Has de Rham filtration.

$\mathcal{L}$ is a free $\mathbb{B}_{dR}^+$-module.

w/ can. isomorphism $\mathcal{L} \otimes_{\mathbb{B}_{dR}^+} \mathbb{C}_p \cong H^i_{dR}(X)$. ($\mathbb{C}_p = \mathbb{B}_{dR}^+/(t)$)

$V \otimes \mathbb{B}_{dR}^+ \subseteq V \otimes \mathbb{B}_{dR} = \mathcal{L}[\frac{1}{t}]$

$\cdots_{dR} = \cdots_{dR} \otimes \cdots$
$\| \mathcal{L}'$ induces a trace filtration on $\mathcal{L} \otimes_{B_{dR}^+} \mathbb{C}_p = H^i_{dR}(X)$ is the Hodge filtration.

$\mathcal{L}$ induces a trace filtration on $(V \otimes B_{dR}^+) \otimes \mathbb{C}_p = V_{\mathbb{C}_p}$: the Hodge-Tate filtration

Lattice is more information.

p-adic HT à la Fontaine: $X/K \leftarrow$ Fin. ext. of $\mathbb{Q}_p$.

$$G_K \circlearrowright H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p) = V$$

True but not easy:

the lattice $\mathcal{L}$ is

$$(V \otimes B_{dR})^{G_K} \otimes B_{dR}^+.$$

$\uparrow$ K-vector space of same dim'n as $H^i_{et}(X, \mathbb{Q}_p)$

$\|$

$D(V) = H^i_{dR}(X) \leftarrow$ K vector space.

has trivial Galois action.

$$V \otimes B_{dR}^+ \subseteq D(V) \otimes B_{dR}$$

$\uparrow$ G_K-eq. lattice

$\nwarrow$ G_K-action all on this factor

$\nwarrow$ determined by Hodge filtration (like $\mathbb{C}^\times$-eq. lattice $\leftrightarrow$ filtration).

If defined$/K$ Hodge filtration determines $V \otimes B_{dR}^+ \subseteq D(V) \otimes B_{dR}$.

X/K $H^i_{dR}(X)$, Fil$^\cdot$ $\rightarrow$ Recover $V \otimes B_{dR}^+$.

Question: Can you remember more structure on $H^i_{dR}(X)$ to recover V itself

Yes: via crystalline cohomology. $\rightsquigarrow$ Canonical $K^{ur} \subseteq K$

vector space
with semilinear Frobenius
underlying H^i_{dR}.

Fontaine's theory about recovering this
via period ring $\mathbb{B}_{crys}$

Rest of today, Thursday:
Explain the internal geometry of period rings governing this
kind of comparison,
admissibility vs weakly admissibility

Right way to organize structures/rings/etc.
appearing p-adic HT.

Idea: $W(\mathcal{O}_{\mathbb{C}_p^\flat}) \sim$ power series with coefficients in $\mathcal{O}_{\mathbb{C}_p^\flat}$,
the variable p.

$\mathcal{O}_{\mathbb{C}_p^\flat}$ is complete for valuation.

$$\mathcal{O}_{\mathbb{C}_p^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$$

$$\ni \quad |(a_0, a_1, a_2, \ldots\ldots)| = |a_0|_p.$$

this an abs. value on $\mathcal{O}_{\mathbb{C}_p^\flat}$.

$$\mathbb{C}_p^\flat = \varprojlim_{x \mapsto x^p} \mathbb{C}_p$$

Extends to
$\mathbb{C}_p^\flat = \mathrm{Frac}\, \mathcal{O}_{\mathbb{C}_p^\flat}$.
$\mathcal{O}_{\mathbb{C}_p^\flat}$ is the valuation ring.

$W(\mathbb{C}_p^\flat) \longleftrightarrow$ like $\mathbb{C}[[t]]$.

purely formal power series
– no convergence condition.

$\bigcup$

$\Sigma a_n x^n$

s.t. $\lim_{n} |a_n^{1/n}| \le \frac{1}{c}$

$\Sigma a_n r^n = 0$

$\forall\, r < c$.

Think of as

functions on $|z| < C$.

$$\sum_{n=0}^{\infty} [a_n]\, p^n$$

Have $|\ |$ on $\mathbb{C}_p^b$

$\rightsquigarrow$ can impose growth conditions on the coefficients

Σ really become functions on a natural p-adic analytic space.