Last time Enriched filtrations by replacing them with lattices in $\mathbb{C}((t))^n$

(Lattice $\mathcal{L} \rightsquigarrow$ trace filtration on $\mathbb{C}^n = \mathbb{C}[[t]]^n/(t)$.)

Important fact! If μ is minuscule then this is a bijection

Lattices of type $\mu \longleftrightarrow$ Filtrations of type μ.

For general μ get bijection

$\mathbb{C}^*$-equivariant lattices of type $\mu \longleftrightarrow$ Filtrations of type μ.

Today Introduce p-adic analog.

Naive guess: $\mathbb{C}_p((t))$ ~ Above is all still true.

Galois action is important ~ ok just use semilinear on $\mathbb{C}_p$.

Not enough: Let's use $G_{\mathbb{Q}_p}$ acting on t by cyclotomic character.

$$\left(\chi_p : G_{\mathbb{Q}_p} \to \underset{\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)}{\overset{\shortparallel}{\mathbb{Z}_p^\times}}\right).$$

In fact still the wrong answer.

This is "apparent after you know the right answer"!

Construct $\mathbb{B}_{dR} = \operatorname{Frac} \mathbb{B}_{dR}^+ = \mathbb{B}_{dR}^+[\frac{1}{t}]$

$$0 \to \mathfrak{m} \to \mathbb{B}_{dR}^+ \twoheadrightarrow \mathbb{C}_p \to 0$$

$\mathfrak{m} = (t)$, complete DVR

$G_{\mathbb{Q}_p}$

transform by cyclo character

Enrich Filtration on $\mathbb{C}_p$-vector spaces via B_{dR}^{+}-lattice in $\hat{B}_{dR}$.

Recall notation:

$$\begin{array}{ccccc} \mathbb{Q}_p & \subseteq & \overline{\mathbb{Q}_p} & \subseteq & \mathbb{C}_p \\ \cup\!| & & \cup\!| & & \cup\!| \\ \mathbb{Z}_p & \subseteq & \overline{\mathbb{Z}_p} & \subseteq & \mathcal{O}_{\mathbb{C}_p} \end{array}$$

$\mathbb{C}_p$ = completion of $\overline{\mathbb{Q}_p}$ for $|\ |_p$.

valuation ring.

$\mathcal{O}_{\mathbb{C}_p} \cong \varprojlim_n \overline{\mathbb{Z}_p}/p^n$.

Goal: Want to produce a complete dvr with residue field $\mathbb{C}_p$ and a compatible natural Galois action.

Step 1: Go to $\mathcal{O}_{\mathbb{C}_p}$ (Changes along w/ because Galois action is continuous),

as topological ring.

as a p-adically complete ring.

Going to use adjunction w/ Witt vectors

Strict p-rings $\longleftrightarrow$ Perfect rings

= p-adically complete w/ perfect residue field

$W(R) \longleftarrow\!\!\!\!\shortmid$ R

$$S \to S/p.$$

$$\mathrm{Hom}(W(R), T) = \mathrm{Hom}(R, T)$$

for R, T perfect.

p-adically complete rings $\longrightarrow$ perfect rings.

$$S \longmapsto S^\flat$$

$$\mathrm{Hom}(W(R), S) = \mathrm{Hom}(R, S^\flat).$$

$\mathcal{O}_{\mathbb{C}_p}$ ← we want a functorial construction of something mapping surjectively.

$\mathcal{O}_{\mathbb{C}_p}/p$ ← Not perfect.

If it were, then we'd get $W(\mathcal{O}_{\mathbb{C}_p}/p) \to \mathcal{O}_{\mathbb{C}_p}$.

Let's replace w/a perfect ring mapping to it

If R is my p-adically complete ring.

$$R^\flat = \varprojlim_{\substack{x \mapsto x^p \\ = \text{Frobenius}}} (R/p)$$

← Universal perfect ring mapping to R/p.

Good thing about $\mathcal{O}_{\mathbb{C}_p}/p$: Frobenius is surjective.

$$\overline{\mathbb{Z}}_p/p$$
$$\mathcal{O}_{\mathbb{C}_p}^{\flat} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}/p$$

↑ $\overline{\mathbb{Q}}_p$ is algebraically closed

(projection onto first element in the limit).

$(a_0, a_1, a_2, \dots) \to a_0.$

Count of adjunction:

$$W(\mathcal{O}_{\mathbb{C}_p}^{\flat}) \to \mathcal{O}_{\mathbb{C}_p}.$$

Note: $\mathcal{O}_{\mathbb{C}_p}^{b} \overset{\text{as sets}}{=} \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$

$$\| \qquad \swarrow \text{mod } p.$$

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p.$$

$(a_0, a_1, \dots) \mapsto (b_0, b_1, \dots).$

$a_i^p = a_{i+1}$

Recall: Raising to p^nth power improves congruences

$$b_0 = \lim \tilde{a}_i^{p^i}$$

↑ in $\mathcal{O}_{\mathbb{C}_p}$, exist because p-adically complete.

where $\tilde{a}_i$ is any lift of a_i.

$$G_{\mathbb{Q}_p} \circlearrowright W(\mathcal{O}_{\mathbb{C}_p}^{\flat}) \xrightarrow{\theta} \mathcal{O}_{\mathbb{C}_p}.$$

← surjective, both sides p-adically complete so suffices to check mod p:

$$\mathcal{O}_{\mathbb{C}_p}^{\flat} \overset{\text{ii}}{\to} \mathcal{O}_{\mathbb{C}_p}/p.$$

$$[(a_0, a_1, \ldots\ldots)] \to b_0 = \lim_{i\to\infty} \tilde{a}_i^{p^i}.$$

<u>Next</u>: Invert p, complete along $\operatorname{Ker}\theta$.

$$B_{dR}^+ = \left(W(\mathcal{O}_{\mathbb{C}_p}^\flat)[\tfrac{1}{p}]\right)^\wedge \leftarrow \operatorname{Ker}\theta\text{-adic completion}$$

$$B_{dR}^+ \twoheadrightarrow \mathbb{C}_p.$$

For B_{dR}^+ to be a complete dvr: need $\operatorname{Ker}\theta$ in $W(\mathcal{O}_{\mathbb{C}_p}^\flat)[\frac{1}{p}]$ is a principal ideal.

Fix a compatible system p^{1/p^n}; $\pi = (p, p^{1/p}, p^{1/p^2}, \ldots)$ in $\mathcal{O}_{\mathbb{C}_p}^\flat$

$$\tau := [\pi] - p$$

$\tau \in \operatorname{Ker}\theta$: $W(\mathcal{O}_{\mathbb{C}_p}^\flat) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}$ $\quad (\theta([\pi]) = p)$

<u>Claim</u>: $\operatorname{Ker}\ W(\mathcal{O}_{\mathbb{C}_p}^\flat) \overset{\theta}{\twoheadrightarrow} \mathcal{O}_{\mathbb{C}_p} \to \mathcal{O}_{\mathbb{C}_p}/p$

(p, τ).

Suppose $\sum_{i=0}^{\infty} [c_i] p^i$ in $\operatorname{Ker}\theta$.

$c_i = (c_{i,0}, c_{i,1}, \ldots)$ in $\lim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} = \mathcal{O}_{\mathbb{C}_p}^\flat$.

$$0 = \theta(\ldots) = \sum c_{i,0}\, p^i$$

get $c_{0,0} \equiv 0 \bmod p$.

$\pi \mid C_0$

$\Downarrow$

Can deduce $\operatorname{Ker}\theta : W(\mathcal{O}_{\mathbb{C}_p}^\flat)[\tfrac{1}{p}] \to \mathbb{C}_p$ is generated by ξ.

<u>Last thing</u> I want a generator t for $\operatorname{Ker}\theta : \mathbb{B}_{dR}^+ \to \mathbb{C}_p$. that transforms via χ_p.

<u>Note</u> this t is <u>only</u> an element of $\mathbb{B}_{dR}^+$, not of $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\tfrac{1}{p}]$.

<u>Exercise</u> Fix $1, \zeta_p, \zeta_{p^2}, \zeta_{p^3} \ldots$ compatible system of primitive p-power roots of unity.

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}_{\mathbb{C}_p}^\flat .$$

$$t = \log[\varepsilon] := ([\varepsilon]-1) - \frac{([\varepsilon]-1)^2}{2} + \frac{([\varepsilon]-1)^3}{3} \ldots$$

This is in $\operatorname{Ker}\theta$ because

$[\varepsilon]-1$ is.

Check:

1) $\ker\theta = (t)$

2) $\sigma(t) = \chi_p(\sigma)t$.

WARNING: for $p>2$
$[\varepsilon]-1$ is <u>not</u> a
generator for $\ker\theta$ in $W(\mathcal{O}^{\flat}_{\mathbb{C}_p})[\frac{1}{p}]$.

In here generators are.
$\sum [c_i]p^i$ in $\ker\theta$
and c_1 should be
a unit in $\mathcal{O}^{\flat}_{\mathbb{C}_p}$.

<u>But</u> $1+[\varepsilon^{1/p}]+[\varepsilon^{1/p}]^2 \cdots [\varepsilon^{1/p}]^{p-1}$

$$= \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}.$$

Brinon-Conrad's notes good source for these computations