Last time Finished HT filtrations on Tate modules + Galois action to give a splitting.

Today we're going to enrich filtrations

$$\mathbb{C}((t)) \qquad \mathbb{C}[[t]]$$
$$\overset{\shortparallel}{K} \qquad \overset{\shortparallel}{\mathcal{O}}$$

$V = \mathbb{R}^n$ $\quad e_1, \dots, e_n$ for std basis.

$V_K = \mathbb{C}((t))^n$ $\quad$ (Subscript = extension of scalars).

$V_{\mathcal{O}} = \mathbb{C}[[t]]^n$

$$V \hookrightarrow V_{\mathcal{O}} \xrightarrow{\text{mod } t} V_{\mathbb{C}}$$

Real Hodge structure = a real vector space V + filtration on $V_{\mathbb{C}}$ $(= V \otimes \mathbb{C})$. satisfying some properties

Suppose $\mathcal{L}$ is a lattice in $V_K = \mathbb{C}((t))^n$.
i.e. free $\mathbb{C}[[t]]$ sub-module with $\mathcal{L}[\frac{1}{t}] = V_K$.
2nd part $\Rightarrow$ of rank n.

Example Suppose V is a vector bundle on $\mathbb{P}^1_{\mathbb{C}}$ and $\phi: \mathcal{O}^n|_{\mathbb{A}^1} \xrightarrow{\sim} V|_{\mathbb{A}^1}$
(meromorphic, algebraic)

$$\mathcal{O}^n\big|_{\text{inf. nbhd of } \infty} \sim \mathbb{C}[[t]]^n$$

$$\mathcal{O}^n\big|_{\text{punctured inf nbhd of } \infty} \sim \mathbb{C}((t))^n$$

$V\big|_{\text{inf nbhd of } \infty}$ = Free $\mathbb{C}[[t]]$ module of rank n.

$$\begin{array}{ccc} V\big|_{\text{inf nbhd of } \infty} & & \\ \cap & & \\ V\big|_{\text{punctured inf nbhd of } \infty} & \xrightarrow[\sim]{\phi^{-1}} & \mathcal{O}^n\big|_{\text{punctured nbhd of } \infty} \\ & & \| \end{array}$$

$$\mathcal{L} = \phi^{-1}\left(V\big|_{\text{inf nbhd of } \infty}\right) \subseteq \mathbb{C}((t))^n$$

<u>Idea:</u> $\mathcal{O}^n$ and V are the same <u>except</u> declared a different set of sections to be holomorphic at ∞

Want to go from $\mathcal{L}$ to a (decreasing) filtration $\mathrm{Fil}^\bullet$ on $V_{\mathbb{C}} = V_{\mathcal{O}}/t$

$$\mathrm{Sp}: V_{\mathcal{O}} \to V_{\mathcal{O}}/t = V_{\mathbb{C}}$$
$$\mathbb{C}[[t]]^n \to \mathbb{C}^n$$

$$\mathrm{Fil}^i V_{\mathbb{C}} = \mathrm{Sp}\left(V_{\mathcal{O}} \cap t^i \mathcal{L}\right).$$

<u>Observation</u> Since $\mathcal{L}$ is a lattice, $V_{\mathcal{O}}$ is a lattice.

$$V_{\mathcal{O}} \subseteq t^i \mathcal{L} \quad \text{for } i \ll 0$$
$$t V_{\mathcal{O}} \supseteq t^i \mathcal{L} \quad \text{for } i \gg 0$$

$$\Longrightarrow \quad \mathrm{Fil}^i V_{\mathbb{C}} = V_{\mathbb{C}} \quad \forall\ i \ll 0$$

$$\mathrm{Fil}^i V_{\mathbb{C}} = \{0\} \quad \forall\ i \gg 0.$$

Example $n=2$ $\quad \mathcal{L} = \mathbb{C}[[t]]e_1 + \mathbb{C}[[t]]te_2$
$(= \langle e_1, te_2 \rangle)$.

If $i \leq -1$ then $e_1 = t^i(t^{-i}e_1)$
$\in V_0 \cap t^i\mathcal{L}$

$(t^{-i}e_1 \in \mathcal{L})$.

$e_2 = t^i(t^{-i}e_2)$
$= t^i(t^{-i-1}te_2)$.
$\in V_0 \cap t^i\mathcal{L}$.

$\mathrm{Fil}^i V_{\mathbb{C}} = V_{\mathbb{C}}. \quad i \leq -1$

$\mathrm{Fil}^0 V_{\mathbb{C}} = \langle e_1 \rangle$

$\mathrm{Fil}^i V_{\mathbb{C}} = 0$ for $i \geq 1$
$t^i\mathcal{L} \subseteq tV_0$.

Observation: $\mathcal{O} = \mathbb{C}[[t]]$ is a PID

elementary divisors for modules

$\Downarrow$

$\exists$ a $\mathbb{C}[[t]]$-module basis for $V_{\mathcal{O}}$ $f_1, \dots, f_n$

s.t. $\mathcal{L} = \langle t^{i_1} f_1, t^{i_2} f_2, \dots, t^{i_n} f_n \rangle$

with $i_1 \leq i_2 \leq i_3 \dots \leq i_n$.

$f_1, \dots, f_n$ are a basis for $V_{\mathcal{O}}$

$\Longrightarrow$ $\overline{f_i} = S_p(f_i)$ are a basis for $V_{\mathbb{C}}$.

Filtration is

$$Fil^j = sp(V_{\mathcal{O}} \cap t^j \mathcal{L})$$

$$= \langle \overline{f_{i_m}} \rangle$$

For $i_m + j \leq 0$.

[illegible]

$\mathbb{C}(t^{\cdot} \; t^{i_n} +_{k}) \ni t_{k}$.

$(i_1, \dots, i_n)$ ⇝ determines the type of the filtration.

$$\begin{pmatrix} t^{i_1} & & \\ & \ddots & \\ & & t^{i_n} \end{pmatrix} = \nu$$

(Map $\mathcal{L}$ of type $\nu \to \mathrm{Fil}^{\bullet}$ of type ν

is a map from Affine Grassmannian $\supset$ type ν cell $\longrightarrow$ (partial) flag variety of type ν)

Bialynicki-Birula map.

Exercise: $\nu = (0, 2)$ $\begin{pmatrix} 1 & \\ & t^2 \end{pmatrix}$

Find two <u>different</u> lattices

$\mathcal{L}_1$ & $\mathcal{L}_2$ of type ν

in $\mathbb{C}((t))^2$ inducing the

Same filtration on $\mathbb{C}^2$.

$\mathcal{L}_1 = \langle e_1, t^2 e_2\rangle \quad \mathcal{L}_2 = \langle e_1 + te_2, t^2 e_2\rangle$

different — $e_1 + te_2$ is not in $\mathcal{L}_1$

e_1, e_2 are a basis for $\mathbb{C}[[t]]^2$

$e_1 + te_2, e_2$ basis for $\mathbb{C}[[t]]^2$.

$\nu = (0,2)$ for both $\mathcal{L}_1$ & $\mathcal{L}_2$.

$sp(e_1) = e_1 \qquad sp(e_2) = e_2$

$sp(e_1 + te_2) = e_1 \qquad sp(e_2) = e_2.$

$\mathrm{Fil}^{-2} = \mathbb{C}^2$

$\mathrm{Fil}^{-1} = \langle e_1\rangle$

$\mathrm{Fil}^{0} = \langle e_1\rangle$

$\mathrm{Fil}^{1} = 0$

$\langle t^{-1}e_1 + e_2, te_2\rangle$

$\Downarrow$

$e_1 = t(t^{-1}e_1 + e_2) - te_2$

$(k, k, k, \ldots, k, k+1, k+1, k+1, \ldots)$

//

Theorem: (1) If ν is minuscule

(then the filtration is just given by a single non-degenerate subspace appearing in exactly one step)

then $\mathcal{L}$ is uniquely determined by $\mathrm{Fil}^\bullet$.

(2) For any ν this gives a bijection

Filtrations of type ν $\longleftrightarrow$ $\mathbb{C}^\times$-equivariant lattices in $\mathbb{C}((t))^n$.

$$\mathbb{C}^\times \circlearrowright \mathbb{C}((t))$$
$$\lambda \cdot f(t) = f(\lambda t).$$

$$\mathbb{C}^\times \circlearrowright \mathbb{C}((t))^n$$

$\mathcal{L}$ is equivariant if $\lambda \cdot \mathcal{L} = \mathcal{L}$ for any $\lambda \in \mathbb{C}^\times$.

Example

$\langle e_1, t^2 e_2 \rangle$

$$\begin{aligned} \lambda \langle e_1, t^2 e_2 \rangle &\\ \parallel &\\ \langle e_1, \lambda^2 t^2 e_2 \rangle &\\ \parallel &\\ \langle e_1, t^2 e_2 \rangle. \end{aligned}$$

$\lambda \langle e_1 + te_2, t^2 e_2 \rangle$

$\langle e_1 + \lambda te_2, \lambda^2 t^2 e_2 \rangle.$

$\lambda = -1$

$\langle e_1 - te_2, t^2 e_2 \rangle.$

Image mod t^2

$\langle e_1 + te_2 \rangle \quad \langle e_1 - te_2 \rangle.$

Not same sub-module of $(\mathbb{C}[[t]]/t^2)^2$.

<u>Remark</u> Lattices which are = the ones where can take the basis already fix

$V_{\mathbb{C}}$.

Next time: Give the p-adic version of $\mathbb{C}((t))$
$\rightsquigarrow$ abstractly $\mathbb{C}_p((t))$.

$G_{\mathbb{Q}_p} \circlearrowright \mathbb{B}_{dR}$
$\uparrow$
Plays the role of $\mathbb{C}^*$.