Last time Elliptic curve $E/\mathbb{C}_p$ with good reduction

(i.e. $y^2 = x^3 + ax + b$, a, b in $\mathcal{O}_{\mathbb{C}_p}$ $\sim$ gives an elliptic curve $/\overline{\mathbb{F}}_p$ mod $\mathfrak{m}$).

Produced a map $T_pE \otimes \mathbb{C}_p \to \mathbb{C}_p$.

(or more canonically: $T_pE \otimes \mathbb{C}_p \twoheadrightarrow \Omega_E \leftarrow$ invariant differential form.)

Weil pairing gives $(\Lambda^2 T_p E = \mathbb{Z}_p \leftarrow T_p \mathbb{G}_m)$.

get

$$0 \to \mathrm{Lie}\, E \otimes \mathbb{C}_p(1) \to T_pE \otimes \mathbb{C}_p \to \Omega_E \otimes \mathbb{C}_p \to 0$$

short exact sequence. $(\Omega_{E^\vee})$

Hodge-Tate filtration. $(\mathbb{C}_p(1) = \mathbb{C}_p \otimes \mathbb{Z}_p(1))$.

Remarks ① Don't need good reduction $\rightleftarrows$ use explicit description of Tate curve.

② If $E/\mathbb{C}$: Hodge filtration

$$0 \to \Omega_E \to H_1(E(\mathbb{C}), \mathbb{C}) \to \mathrm{Lie}\, E \to 0$$
$$H_1(E(\mathbb{C}), \mathbb{C}) = H_1(E(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}.$$

Should really be $\Omega_{E^\vee}$.

③ If E/K, $K/\mathbb{Q}_p$ fin. extension of $\mathbb{Q}_p$, everything is Galois-equivariant. (action of G_K).

Important: In setting of ③, the sequence is of $\mathbb{C}_p$-semilinear representations of G_K

$\mathbb{C}_p$ vector space V

$G_K \circlearrowright V$ Normal rep: $\rho(\sigma)(a v) = a \cdot \rho(\sigma) \cdot v$ $\quad a \in \mathbb{C}_p$, $v \in V$

Semi-linear rep: $\rho(\sigma)(av) = \sigma(a)\rho(\sigma)\cdot v$

Remark — Semilinear rep $\leftrightarrow$ equivariant vector bundle.

<u>Theorem (Tate)</u>: In the setting of (3) above there is a unique splitting of the Hodge–Tate sequence that is Galois-equivariant i.e. there is a unique G_K-equivariant splitting.

$$T_pE \otimes \mathbb{C}_p = \mathrm{Lie}\, E \otimes_K \mathbb{C}_p(1) \oplus \Omega_E \otimes_K \mathbb{C}_p$$

Proof is purely Galois-theoretic. $\rightsquigarrow$ If you fix a basis for Lie E, Ω_E. then HT sequence is

$$0 \to \mathbb{C}_p(1) \to T_pE \otimes \mathbb{C}_p \to \mathbb{C}_p \to 0$$

in $\mathrm{Ext}_{G_K}(\mathbb{C}_p, \mathbb{C}_p(1))$.

Tate shows $= 0$.

$$\mathrm{Hom}_{G_K}(\mathbb{C}_p, \mathbb{C}_p(1)) = 0.$$

Purely about $\mathbb{C}_p$-semilinear representations of G_K.

These are actually pretty simple in some ways. $\rightsquigarrow$ Can basically replace

G_K with $G_{K(\zeta_{p^\infty})/K} \subseteq \mathbb{Z}_p^\times$.

$\rightsquigarrow$

Tate: p-divisible groups.

$\rightsquigarrow$

This computation of ét groups.

① Used some version of almost purity

$$\mathbb{Q}_p^{cyc} = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$$ is a perfectoid field.

$\Updownarrow$

This is what makes perfectoid algebras so special.

In a precise sense / for the measures we discussed (differentials / discriminants / trace pairings)

almost all ramification in going from $\mathbb{Q}_p$ to $\overline{\mathbb{Q}}_p$ has already occurred in going from $\mathbb{Q}_p$ to $\mathbb{Q}_p^{cyc}$.

E.g. $L/\mathbb{Q}_p^{cyc}$ is a finite extension

then $\Omega_{\mathcal{O}_L/\mathcal{O}_{\mathbb{Q}_p^{cyc}}}$ ← annihilated by $\mathfrak{m}$ in $\mathcal{O}_L$.

contains elements z $|z|$ arbitrarily close to 1 ($v_p(z)$ arbitrarily close to zero)

② Once $\mathbb{Z}_p^\times \supseteq 1 + p\mathbb{Z}_p$

basically a cyclic group.

Very explicit in terms of cocycles

Need Tate's normalized trace,

Normally $tr : L \to K$ if L/K is a finite extension

Tate: $tr : \mathbb{Q}_p^{cyc} \to \mathbb{Q}_p$

continuous

$$tr|_{\mathbb{Q}_p(\zeta_{p^n})} = \frac{tr_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}}{p^n}.$$

Remark All of this extends to

1. Abelian varieties
2. p-divisible group.
3. Cohomology groups of smooth proper rigid analytic spaces.

Filtrations have more steps in higher cohomology groups.

Filtrations are the wrong structure to consider.

$$\mathbb{C}, \quad \mathbb{C}[[t]], \quad \mathbb{C}((t)).$$

Let V be an n-dim'l $\mathbb{C}$ vector space.

Richer structure on V that gives rise to a filtration:

Filtration on V:

$$V \supseteq \cdots \supseteq \mathrm{Fil}^i \supseteq \mathrm{Fil}^{i+1} \supseteq \mathrm{Fil}^{i+2} \supseteq \cdots \supseteq 0$$

with $\mathrm{Fil}^i = V$ for $i \ll 0$,

and $\mathrm{Fil}^i = 0$ for $i \gg 0$.

E.g. $H_1(E(\mathbb{C}), \mathbb{C}) = V$

$\cdots \mathrm{Fil}^{-1} = V \quad \mathrm{Fil}^0 = \Omega_E \quad \mathrm{Fil}^1 = 0 \cdots$

Hodge filtration

$\boxed{\text{Goal}}$ ← Next time.

Exercise: Compute $\Omega_{\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{Z}_p[\zeta_p]}$

$\mathcal{O}_{\mathbb{Q}_p(\zeta_{p^n})}$ (Valuation ring in $\mathbb{Q}_p(\zeta_{p^n})$), $\mathcal{O}_{\mathbb{Q}_p(\zeta_p)}$.

$$\wr\!\!\parallel$$

$$\mathbb{Z}_p[\zeta_{p^n}]/I$$

$I = (a)$ what is $|a|$?

$M_\zeta(x) = x^{p^{n-1}} - \zeta_p \qquad S = \mathbb{Z}_p[\zeta_{p^n}]$

$R = \mathbb{Z}_p[\zeta_p]$

$p^{n-1} x^{p^{n-1}-1}$

$\Omega_{\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{Z}_p[\zeta_p]}$

$S = R[x]/f(x)$

$(f'(x)) = I.$

$\Omega_{R[x]/m_{\zeta_{p^n}}(x)/R}$

$= S\,dx \big/ M'_{\zeta_{p^n}}(x)\,dx.$

generators annihilator

$(p^{n-1}\zeta_{p^n}^{p^{n-1}-1}) = (p^{n-1}).$

$|p^{n-1}| = \frac{1}{p^{n-1}}.$

$\Omega_{\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{Z}_p}$

divisible $\mathbb{Z}_p[\zeta_{p^n}]$ module.

$$x \to x^p$$