Last time Representations from elliptic curves (Tate module).

E/K an elliptic curve

$K/\mathbb{Q}_p$ finite extension.

$$\overset{\text{rank 2 } \mathbb{Z}_\ell\text{-module}}{T_\ell E} \circlearrowleft G_K$$

$$\underset{\text{rank 2 } \mathbb{Q}_\ell\text{-vector space}}{V_\ell E}$$

Tate curve $E_\pi(\mathbb{C}_p) \cong \mathbb{C}_p^\times/\pi^{\mathbb{Z}}$.

$(|\pi|<1)$

$$T_\ell E = \begin{pmatrix} \underset{\ell\text{-adic cyclotomic}}{\chi_\ell} & \underset{\text{from roots of } \pi}{\tau_{\pi,\ell}} \\ 0 & 1 \end{pmatrix}$$

$\ell \neq p$ tamely ramified.

$\ell = p$ wildly ramified $\Big\}$ Kummer theory

Started on another example.

Example: $K = \mathbb{Q}_9$ = degree 2 unramified extension of $\mathbb{Q}_3$.

$i \in K$ a square root of -1.

$E: y^2 = x^3 - x$

Observation: $f_i : E \to E$

$$(x,y) \mapsto (-x, iy)$$

$$f_i \circ f_i = (x,y) \mapsto (x,-y)$$
$$\overset{\shortparallel}{-1} \text{ in group law on } E.$$

Defined by polynomial w/ coeff. in K.

$\mathrm{End}(E)$ =: Endomorphisms of E as an algebraic variety & group.

$\mathrm{End}(E) \supseteq \mathbb{Z}$, $\mathrm{End}(E) \ni f_i$

$\mathbb{Z}[f_i] \cong \mathbb{Z}[x]/(x^2+1)$

$$\mathrm{End}(E) \circlearrowright T_\ell E \circlearrowleft G_K$$

$$\boxed{V_\ell E}$$

$$\mathrm{End}(E)\otimes \mathbb{Q}_\ell \circlearrowright V_\ell E \circlearrowleft G_K$$

$$\begin{array}{c} \cup \\ \mathbb{Q}_\ell[x]/x^2+1 \end{array}$$

These actions commute

A priori: G_K acts on $V_\ell E$ which is a 2-dim'l $\mathbb{Q}_\ell$-vector space.

Now: G_K acts on $V_\ell E$ which is a module over $\mathbb{Q}_\ell[x]/x^2+1$.

$\ell = p = 3$	$\ell = 5$	$\ell = 7$
$\mathbb{Q}_\ell[x]/x^2+1 = \mathbb{Q}_9$	$\mathbb{Q}_5[x]/x^2+1 \cong \mathbb{Q}_5 \times \mathbb{Q}_5$	$\mathbb{Q}_{49}$.

In all cases, $V_\ell E$ is a free rank 1 module

$G_K \to GL_2(\mathbb{Q}_\ell)$

$\ell = p = 3$	$\ell = 5$	$\ell = 7$.
$G_K \to GL_1(\mathbb{Q}_9)$		
$G_{\mathbb{Q}_9} \to \mathbb{Q}_9^\times$	$G_{\mathbb{Q}_9} \to \mathbb{Q}_5^\times \times \mathbb{Q}_5^\times \subset GL_2(\mathbb{Q}_5)$ via diagonal matrices	$G_{\mathbb{Q}_9} \to \mathbb{Q}_{49}^\times$.

(Note: these are all abelian!).

$G_K^{ab} = \hat{\mathbb{Z}} \times \mathbb{Z}_9^\times$ (local class field theory). not ramified.

$\langle \mathrm{Frob} \rangle$ factors through $\hat{\mathbb{Z}}$ for $\ell = 5, 7$

for $\ell = p = 3$ it's basically the obvious map.

$$G_K^{ab} \to \mathbb{Z}_q^\times \hookrightarrow \mathbb{Q}_q^\times \quad \leftarrow \text{ very wildly ramified}$$

<u>Definition.</u> $K/\mathbb{Q}_p$ ($p \neq 2$) fin. extension $\mathcal{O}_K$ valuation ring.

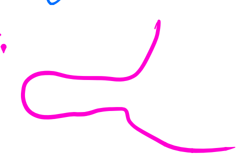
E/K an elliptic curve has good reduction if $\exists$ equation $y^2 = x^3 + ax^2 + bx + c$ for E with $a, b, c \in \mathcal{O}_K$

s.t. $y^2 = x^3 + \bar{a}x^2 + \bar{b}x + \bar{c}$

is an elliptic curve over $\mathfrak{K} \leftarrow$ residue field

$= \mathbb{F}_q$.

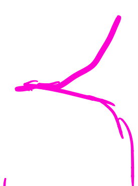
$y^2 = f(x)$ tangent line at every point if $f(x)$ has no multiple roots.

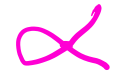


<u>Example</u> $y^2 = x^3 + 3$ over $\mathbb{Q}_3$

reduce mod 3 get

$y^2 = x^3$



(Tate curves have bad reduction $\rightsquigarrow$  mod $\mathfrak{m}$.)

<u>Example</u> $y^2 = x^3 - x$ has good reduction over K if $K/\mathbb{Q}_p$ $p \neq 2$.

$x^3 - x = x(x+1)(x-1)$ $\quad 1, -1, 0$ are distinct in $\mathbb{F}_p$ for $p \neq 2$.

<u>Fact:</u> If E/K ($K/\mathbb{Q}_p$) has good reduction $\bar{E}/\mathfrak{K}$.

$T_\ell E \xrightarrow{\sim} T_\ell \bar{E}$ for any $\ell \neq p$.

(think Hensel's lemma).

$$\begin{tikzcd} G_K \arrow[d, two heads] & \circlearrowright & T_\ell E \arrow[d, phantom, "\wr\|"] \\ G_k & \circlearrowright & T_\ell \bar{E} \end{tikzcd}$$

$G_K \circlearrowright T_\ell E$ is unramified.

Example $E: y^2 = x^3 - x$ $\quad K = \mathbb{Q}_q$

saw $T_\ell E$ gave unramified for $\ell \neq p$.

Tate curve: saw $T_\ell E$ is ramified for $\ell \neq p$.

has bad reduction.

Fact (Néron-Ogg-Shafarevich): E has good reduction $\iff$ $T_\ell E$ is unramified for $\ell \neq p$. (for any / all).

ℓ-adic étale cohomology for $\ell \neq p$ is too simple to say much if you have good reduction

$\ell = p$ get ramified representations both for good and bad reduction.

One motivation for p-adic HT: understand how to distinguish good reduction from bad reduction using p-adic

Tate module.

crystalline

(also want to understand which representations come from algebraic varieties at all).

Fontaine-Mazur.

generic — de Rham representation

p-adic HT à la Fontaine.

crystalline — good reduction

Semistable — Tate curve.

Do something that is p-adic HT but doesn't need a Galois representation. (i.e. we can use $T_p E$ for $E/\mathbb{C}_p$ an elliptic curve, not E/K $K/\mathbb{Q}_p$ finite extension).

We'll find a 1-dim'l subspace of $V_p E \otimes_{\mathbb{Q}_p} \mathbb{C}_p$

(like Hodge-Theory $H_1(E(\mathbb{C}), \mathbb{Q})$, 2 dim'l $\mathbb{Q}$-vector space

one-dim'l subspace $H_1(E(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}$)

<u>Definition/Fact/Recall:</u> K of characteristic 0

E/K is an elliptic curve. For any n.

$$E[n](\bar{K}) \times E[n](\bar{K}) \to \mu_n(\bar{K}).$$

(Weil pairings). non-degenerate antisymmetric. (like polarization).

Compatible, Galois-compatible,

$$V_\ell E \times V_\ell E \to \mathbb{Q}_\ell(1).$$

non-degenerate antisymmetric pairing.

$$(\Longrightarrow) \quad \det V_\ell E = \Lambda^2 V_\ell E \cong \mathbb{Q}_\ell(1)$$

as a G_K-representation),

Upgrade this to get our filtration.

(Work over $\mathcal{O}_K$).