Last time: Showed if $K/\mathbb{Q}_p$ is a fin extension and $\ell \neq p$, then any $\rho : \underset{\mathrm{Gal}(\overline{K}/K)}{\overset{\shortparallel}{G_K}} \to GL_n(\mathbb{Q}_\ell)$

satisfies: $\exists$ K'/K finite such that $\rho|_{G_{K'}}$ is tamely ramified.

Recall

$$G_{K'} \to G_{K'}^{tr} \cong \hat{\mathbb{Z}}^{(p)}(1) \rtimes \hat{\mathbb{Z}}$$

φ Frob q — $q = p^s$

Fix a compatible system of roots of unity $\quad \cong$

(prime-to-p)

$$\prod_{\substack{r \text{ prime} \\ r \neq p}} \mathbb{Z}_r \rtimes \hat{\mathbb{Z}}$$

$\tau = (1, 1, \ldots, 1)$. $\quad \sigma = \mathrm{Frob}_q$

$$\boxed{\sigma \tau \sigma^{-1} = q\tau}$$

$\rho|_{G_{K'}}$ factors through $\mathbb{Z}_\ell \rtimes \hat{\mathbb{Z}}$

(because by construction image is a pro-ℓ-group).

Exercise: The eigenvalues of $\rho(\tau)$ are all ℓ-power roots of unity.

Observation $\quad \rho(\sigma\tau\sigma^{-1}) = \rho(\sigma)\rho(\tau)\rho(\sigma)^{-1}$ has same e.v. as $\rho(\tau)$ (conjugate matrix)

$\rho(\sigma\tau\sigma^{-1}) = \rho(\tau)^q$ $\quad (\sigma\tau\sigma^{-1} = q\tau)$.

Eigenvalue of $\rho(\tau)^q$ = qth power of eigenvalues of ρ.

$\{\text{E.v. of } \rho(\tau)\}$ maps into itself by taking qth powers.

Consequence $\rho|_{G_{K''}}$ K''/K' finite.

such that ρ(top. generator of tame inertia) is unipotent.

Consequence $\rho|_{G_{K''}}$ determined by $\rho(\sigma)$ and a nilpotent matrix. "$\log$"$(\rho(\tau))$.

$\rightsquigarrow \rho$ determined by a representation r with finite image on inertia + Nilpotent matrix N. (satisfying some properties).

$\Longrightarrow$ Representations of G_K on $\mathbb{Q}_\ell$-vector spaces (continuous)
$\longleftrightarrow$ Finite on inertia representation + Nilpotent matrix.

can compare for $\ell \neq \ell' = p$.
(Weil-Deligne rep.)

(Good source: Toby Gee AWS notes)

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p-adic representation of G_K ($K/\mathbb{Q}_p$ finite).
($\rho: G_K \to GL_n(\mathbb{Q}_p)$ continuous rep'n).

$1 + pM_n(\mathbb{Z}_p)$ is a pro-p group.
$\rho|_{P_K}$ can be very complicated.
($\mathrm{Map}_{cont}(P_K, 1 + pM_n(\mathbb{Z}_p))$
$= (1 + pM_n(\mathbb{Z}_p))$ countable set.

$(\ldots)$ generators of P_K as a free pro-p group

<u>Note</u>: There are relations imposed by saying ρ comes from restriction from G_K to P_K

so not as big of a problem but still big.

More complicated story!

Examples of p-adic representations of G_K:

$\mathcal{O}_K = R_K$. π uniformizer

① Abelian representations.

$$G_K^{ab} \cong \mathcal{O}_K^\times \times \hat{\mathbb{Z}}$$

Inertia $\nearrow \mathcal{O}_K^\times$, $\hat{\mathbb{Z}} \uparrow \langle \mathrm{Frob}_q \rangle$

$$\| $$

$$1+\pi\mathcal{O}_K \times \mathbb{F}(K)^\times \times \hat{\mathbb{Z}}$$

wildly ramified, tamely ramified, $\langle \mathrm{Frob}_q \rangle$ unramified.

<u>Example</u>

$K \cong \mathbb{Q}_p^{[K:\mathbb{Q}_p]}$ as a $\mathbb{Q}_p$-vector space.

$$\mathcal{O}_K^\times \circlearrowright K \cong \mathbb{Q}_p^{[K:\mathbb{Q}_p]}$$

Frob_q acts trivially.

② Cyclotomic character

$$\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) \overset{\text{can}}{\cong} (\mathbb{Z}/p^n\mathbb{Z})^\times.$$

$$\chi_{p^n}: G_{\mathbb{Q}_p} \to \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$$

$$G_{\mathbb{Q}_p} \to \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times \cong \mathbb{Z}_p^\times$$

$$\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \qquad \cap$$

$$\chi : G_{\mathbb{Q}_p} \xrightarrow[\text{p-adic cyclotomic character}]{} GL_1(\mathbb{Q}_p)$$

$$G_{\mathbb{Q}_p} \circlearrowright \mathbb{Z}_p(1) := \varprojlim \mu_{p^n}(\overline{\mathbb{Q}_p})$$

$\mathbb{Z}_p$ - module

Free of rank 1.

(compatible system of roots of unity = basis element).

Matrix of this representation in any basis

is χ

③ Kummer extensions. (Look like totally ramified extensions)

$\mathbb{Q}_p(\zeta_{p^\infty})$ Fix $a \in \mathbb{Q}_p$ not a root of unity.

$\mathbb{Q}_p(\zeta_{p^\infty}, a^{1/p^\infty})$. $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbb{Q}_p)$.

$$\cong$$

$$\underset{\text{Kummer theory}}{\mathbb{Z}_p(1)} \rtimes \underset{\chi}{\mathbb{Z}_p^\times}$$

$\boxed{\text{Different } a \text{ will lead to different extensions generally.}}$

$(\mathbb{Q}_p^\times)/(\mathbb{Q}_p^\times)^p$. vs. $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^\ell$.

$$\mathbb{Z}_p(1) \rtimes \mathbb{Z}_p^\times \longrightarrow$$

$$\cong$$

Fix a generator

$$\mathbb{Z}_p \rtimes \mathbb{Z}_p^\times \longrightarrow GL_2(\mathbb{Q}_p)$$

$$(a, b) \longrightarrow (\;\; a \;\;)$$

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}.$$

<u>Next time</u>: Elliptic curves, examples coming from elliptic curves.