<u>Know now:</u> Lots about finite extensions of complete discretely valued fields.
$\rightsquigarrow$ a lot about Galois theory (and some about its relation with arithmetic.)

$K/\mathbb{Q}_p$ finite extension.

$$1 \to I_K \to G_K \to G_{k(K)} \to 1$$

$$G_{k(K)} = G_{\mathbb{F}_q} = \langle \mathrm{Frob}_q \rangle$$

$\langle \mathrm{Frob}_q \rangle$ ← topologically generated.

$P_K \trianglelefteq I_K$

$$P \backslash G_K = \hat{\mathbb{Z}}^{(p)}(1) \rtimes \langle \mathrm{Frob}_q \rangle, \qquad \langle \mathrm{Frob}_q \rangle = \hat{\mathbb{Z}}$$

$$\prod_{\ell \neq p} \mathbb{Z}_\ell \cong \hat{\mathbb{Z}}^{(p)} \cong \hat{\mathbb{Z}}^{(p)}(1)$$

after fixing a compatible choice of prime to p- roots of unity

$$\prod_{\ell \neq p} \mathbb{Z}_\ell \rtimes \hat{\mathbb{Z}}$$

1 ↔ compatible choice of roots of unity.

$\sigma \leftrightarrow \mathrm{Frob}_q$

conjugation action

$$\sigma a \sigma^{-1} = q a \quad \text{for } a \in \prod_{\ell \neq p} \mathbb{Z}_\ell.$$

Moreover, this description is built out of arithmetic $\rightsquigarrow$ explicitly constructed maximal tamely ramified extension.

$P \cong$ free pro-p group on countably many generators only an abstract description.

Remark: One more piece we can describe explicitly:

$$G_K^{ab} = R_K^\times \times \underset{\langle \mathrm{Frob}\rangle}{\hat{\mathbb{Z}}}$$

Via local class field theory. $K = \mathbb{Q}_p$ $\quad \mathbb{Z}_p^\times = R_K^\times$

$\mathbb{Q}_p^{ab} = \mathbb{Q}_p(\zeta_N)$ (including p-power roots).

$\mathbb{Z}_p^\times \supseteq \underbrace{1 + p\mathbb{Z}_p}_{\text{wild inertia}}$

Representations:

Understand groups via their actions.

Galois groups are given with an action.

$G_{\mathbb{Q}_p} \circlearrowright \overline{\mathbb{Q}_p} \qquad \circlearrowright \mathbb{C}_p.$

$\circlearrowright$ roots of f for $f \in \mathbb{Q}_p[x]$

Obvious $\underline{\text{things}}$ $G_{\mathbb{Q}_p}$ acts on. ← set, vector space, manifold

M is a manifold. $\mathrm{Diff}(M) \circlearrowright M.$

$\mathrm{Diff}(M) \circlearrowright H^i(M, \mathbb{Z}).$ ← $\mathbb{Z}$-module

$\mathrm{Diff}(M) \circlearrowright \pi_1(M)$ ← fundamental groupoid

If K is $\underline{\text{any}}$ field, $\overline{K}$ is an algebraic closure.

$G_K \circlearrowright \overline{K}$ $\quad (G_K := \mathrm{Aut}(\overline{K}/K)).$

Finite Galois theory ⇝ (with K) for $f \in K[x]$.

$G_K \circlearrowright \mathrm{Roots}(f)$.

$$\mathrm{Roots}(f) = \underline{\underline{V}}(f) \subseteq \mathbb{A}^1.$$

$$\mathrm{Roots}(f) \otimes \mathbb{Q}_\ell = H_0(V(f), \mathbb{Q}_\ell).$$

Any algebraic variety over K

⇝ can get $\mathbb{Q}_\ell$ vector spaces with

a (continuous) action of G_K via étale cohomology.

Important problem in NT: understand these representations.

Need to understand continuous representations of G_K on a $\mathbb{Q}_\ell$ vector spaces.

K = Finite extension of $\mathbb{Q}_p$.

$\ell = p \leftarrow$ complicated, p-adic HT

$\ell \neq p \leftarrow$ pretty simple.

G_K is a profinite group = topological group s.t.

$G_K \cong \lim\limits_{N \text{ open normal subgroups}} G_K/N$

topological

$G_K \cong \lim\limits_{i \in I} G_i$

cofiltered limit of

$\mathbb{Q}_\ell$ topologized using $|\ |_\ell$

$\mathbb{Z}_\ell \in \mathbb{Q}_\ell$

profinite group

$\mathbb{Z}_\ell \cong \lim \mathbb{Z}/\ell^n\mathbb{Z}$.

Finite groups.

Definition: A profinite group G is pro-p for a prime p if

If any quotient by an open normal subgroup is a finite p-group.

$$\iff G \cong \varprojlim_{i \in I} G_i \qquad G_i \text{ finite } p\text{-groups.}$$

Exercise: If G_1 is pro-p and G_2 is pro-ℓ for $\ell \neq p$ then

$$\mathrm{Hom}_{\substack{\text{top} \\ \text{group}}}(G_1, G_2) = 0.$$

Sketch: Suppose f is such a map.

$$f : G_1 \to G_2$$

is uniquely determined by

$$\begin{tikzcd} G_1 \arrow[r] \arrow[rr, bend right, "f_N"'] & G_2 \arrow[r] & G_2/N \end{tikzcd}$$

for all open normal N in G_2.

(definition of a limit).

$\ker f_N = f^{-1}(N)$ which is open by continuity

and thus an open normal subgroup of G_1.

$$f_N : G_1 \to G_1/\ker f_N \hookrightarrow G_2/N$$

finite p-group. finite ℓ-group

so $f_N = 0 \Rightarrow f = 0$.

top. $\cong$ an open subset of $M_n(\mathbb{Q}_\ell)$ $\cong \mathbb{Q}_\ell^{n^2}$

Lemma Let $\rho : G_K \to GL_n(\mathbb{Q}_\ell)$ ($K/\mathbb{Q}_p$ finite, $\ell \neq p$) be a continuous homomorphism. Then there is a finite extension L/K s.t. $\rho|_{G_L}$ factors through $G_L^{\text{tame}} = P_L \backslash G_L$.

Proof: Step 1 - Assume $\rho : G_K \to GL_n(\mathbb{Z}_\ell)$.

(exercise to check this).

Then $1 \to 1 + \ell M_n(\mathbb{Z}_\ell) \to GL_n(\mathbb{Z}_\ell) \to GL_n(\mathbb{F}_\ell) \to 1$

$n = 1$

$$\left(1 \to 1 + \ell \mathbb{Z}_\ell \to \mathbb{Z}_\ell^\times \to \mathbb{F}_\ell^\times \to 1\right)$$

open pro-ℓ subgroup. $\left(1 + \ell M_n(\mathbb{Z}_\ell) = \varprojlim_n \frac{1 + \ell M_n(\mathbb{Z}_\ell)}{1 + \ell^n M_n(\mathbb{Z}_\ell)}\right)$

$\rho^{-1}(1 + \ell M_n(\mathbb{Z}_\ell))$ is an open subgroup U of G_K.

so $U = G_L$ for finite extension of K.

$$\rho|_{G_L} \longrightarrow 1 + \ell M_n(\mathbb{Z}_\ell)$$

Need to show $\rho|_{P_L}$ is trivial.

$$\rho|_{P_L} : \underset{\text{pro-}p\text{ group}}{P_L} \longrightarrow \underset{\text{pro-}\ell\text{ group}}{1 + \ell M_n(\mathbb{Z}_\ell)}.$$

By exercise we are done.