What we've done: Describe a lot of the structure of finite extension of $\mathbb{Q}_p$ / Galois group of finite extensions of $\mathbb{Q}_p$ (and to a lesser extent $\mathbb{C}((t))$, $\mathbb{F}_p((t))$, $W(\bar{k})$, etc. ⟵ complete discretely valued fields).

In particular identified 2 key points:

1) Absolute value extends uniquely to any algebraic extension
2) Can use this to study ramification.

Today: 1) Slight wrinkle in our picture:

Infinite algebraic extensions of $\mathbb{Q}_p$ are not complete

Example: (1) $\mathbb{Q}_p^{ur} = \mathbb{Q}_p(\zeta_N \mid (N,p)=1)$.

Not complete:

$$\sum_{(n,p)=1}^{\infty} \zeta_n p^n$$

does not converge even though it "should"

(2) $\mathbb{Q}_p(p^{\frac{1}{N}}, N \in \mathbb{Z})$.

$$\sum_{n=1}^{\infty} p^{n+\frac{1}{n}}$$

does not converge even though it "should".

Address this by explaining what happens when we complete.

2) Measures of ramification

Infinite complete Galois theory.

$\mathbb{Q}_p \subseteq \overline{\mathbb{Q}_p}$. $\mathbb{C}_p :=$ completion of $\overline{\mathbb{Q}_p}$ for $|\ |_p$.

$\mathbb{C}_p$ is complete and algebraically closed.

by definition

not immediately obvious.

Krasner's lemma gives this.

Two polynomials sufficiently close generate the same splitting field.

$| \ |_p$ extends by continuity to $\mathbb{C}_p$.

so $(\mathbb{C}_p, | \ |_p)$ is a complete valued field.

$\mathrm{Aut}(\mathbb{C}_p/\mathbb{Q}_p) =$ enormous. $\supseteq \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$

Field theoretic automorphisms

extend to $\mathbb{C}_p$ by continuity (they preserve $| \ |_p$).

In fact: $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) = \mathrm{Aut}_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$.

$$\left(\mathrm{Aut}(\mathbb{C}_p/\mathbb{Q}_p) \xrightarrow{\text{by restriction}} \mathrm{Aut}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \right)$$

by continuity.

Aut_{cont} isn't bigger because continuous map is determined by its action on the dense set $\overline{\mathbb{Q}_p}$.).

Theorem: The assignment

Closed subgroups of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) = \mathrm{Aut}_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$ — Complete subfields of $\mathbb{C}_p$

$$H \longmapsto \mathbb{C}_p^H = \{x \in \mathbb{C}_p,\ h(x) = x \ \forall h \in H\}$$

$$\mathrm{Aut}_{cont}(\mathbb{C}_p/K) \longleftarrow\!\shortmid K$$

is an inclusion reversing bijection.

= Completion of

Proof: Use Ax–Sen–Tate lemma.

$\mathbb{Q}_p^H$.

Idea: If $\alpha \in \overline{\mathbb{Q}}_p$, ε in $\mathbb{C}_p$ is small, then $\alpha + \varepsilon$ is close to all of its conjugates under $\mathrm{Aut}_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$.

Ax–Sen–Tate is a converse to this, says: if an element is close to all of its conjugates then it's actually close to something in $\mathbb{Q}_p$.

(Good reference: Fontaine–Ouyang – p-adic Galois representations).

Remark: Worked over $\mathbb{Q}_p$ only for convenience / familiarity. Same theorem is true for $\mathbb{Q}_p$ replaced w/ any K complete non-archimedean valued of mixed characteristic or characteristic p and perfect.

$\mathbb{C}_p$ with $\hat{\overline{K}}$ ($\leftarrow$ completion, $\overline{K}$: sep. closure) $\left(x^p - a \sim x^p + p^{1000}x - a \right)$

E.g. replace $\mathbb{F}_p((t))$ w/ $\mathbb{F}_p((t^{1/p^\infty}))$ — Have same Galois group

---

2. Measures of ramification:

1. Using Galois groups $\sim$ $I \supseteq P = P_1 \supseteq P_2 \supseteq \dots$

   L/K finite Galois extension

   Amount of ramification $\sim$ how deep $P_i \neq 0$.

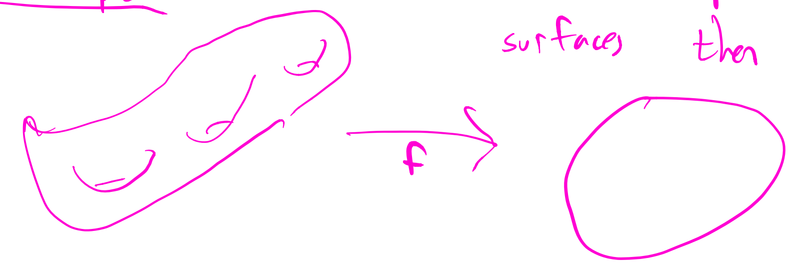
2. Some equivalent ways (for K complete discretely valued)

different, discriminant, differentials. ← we'll look at these

different refines

both defined using the trace pairing.

Example If $f: X \to Y$ is a non-constant map of compact Riemann surfaces then



then for any $z \in X$ $f \underset{\text{near } z}{\sim} z \mapsto z^n$ for some n.

Ramification points are those where $n \geq 2$, (where not a local isomorphism).

If $x_0 \in X$ $f(x_0) \in Y$. t is a coordinate near $f(x_0)$
z is a coordinate near x_0

$$\mathbb{C}[[z]] \longleftarrow \mathbb{C}[[t]]$$
$$h(f(z)). \longleftarrow h(t)$$
$$z^n \longleftarrow t$$

in adapted coordinates

What happens when we pullback differential forms?

Outside of ramification locus pulling back differentials hits everything.

$dz = dz.$ (if $z \mapsto z$ i.e. $n=1$).

Ramified point: $z \mapsto z^n = t$ $n \geq 2$

$$dt \to dz^n = nz^{n-1}dz.$$

has a zero at $z=0$.
of higher & higher
order as n gets bigger.

Differentials can be phrased purely algebraically.

If $R \to S$ is a map of rings

$$\Omega_{R\to S} = \langle ds \quad s\in S \rangle \Big/ \begin{matrix} ds_1s_2 = s_1 ds_2 + s_2 ds_1 \\ dr = 0 \\ \text{for } r\in R. \end{matrix}$$

$$\mathbb{C}[[t]] \to \mathbb{C}[[z]]$$
$$\qquad\qquad \| $$
$$\mathbb{C}[[t]][z]/z^n - t.$$

$$\Omega_{\mathbb{C}[[t]]\to\mathbb{C}[[z]]} = S\,dz \big/ n z^{n-1} dz.$$

$$\cancel{\sim \quad d(z^n - t) = n z^{n-1} dz}$$

$$\mathbb{C}[[z]] \big/ \boxed{z^{n-1}} \quad \text{(generated by } dz\text{)}.$$

(In general if $S = R[x]/f(x)$ ← monic

$\Omega_{R\to S} \cong R[x]/f'(x)$

generated by dx.)

Exercise: Compute differentials for $\mathbb{Z}_3[3^{1/3}]$ over $\mathbb{Z}_3$

$$\mathbb{Z}_3 \;\; \text{over} \;\; \overset{S}{\underset{}{\|}}$$

$\mathbb{Z}_3$ over $\mathbb{Z}_3[3^{1/3}]$, $\| = R$

$\mathbb{Z}_3[\zeta_3]$

$\mathbb{Z}_3[3^{1/3}]$

$\mathbb{Z}_3[\zeta_3]$

$\mathbb{Z}_3[\zeta_9]$.

$$\mathbb{Z}_3[3^{1/5}] = \mathbb{Z}_3[x]/x^5-3$$

Ω ramified

$$(x^5-3)' = 5x^4$$

$$\mathbb{Z}_3[3^{1/5}]/3^{4/5} \longleftarrow \mathbb{Z}_3[x]/(x^5-3,\ 5x^4). \neq 0$$

mod 3 get $\mathbb{F}_3[x]/x^4$.

$$\mathbb{Z}_3[\zeta_5] = \mathbb{Z}_3[x]/x^5-1$$

unramified

$$(x^5-1)' = 5x^4$$

$$\mathbb{Z}_3[x]/(x^5-1,\ x^4) = 0.$$

$$\mathbb{Z}_3[3^{1/3}] = \mathbb{Z}_3[x]/x^3-3$$

$$(x^3-3)' = 3x^2$$

$$\mathbb{Z}_3[3^{1/3}]/3^{5/3}$$