Last time: Computations in Kummer theory. Sloppy proof of maximal tamely ramified extension = what we expect.

Exercise: If $K/\mathbb{Q}_p$ finite extension

$\pi \in K$ uniformizer, R_K valuation ring.

$$K^\times = \pi^{\mathbb{Z}} \times R_K^\times$$
$$= \pi^{\mathbb{Z}} \times k(K)^\times \times 1 + \pi R_K$$

maximal ideal

$$k(K)^\times = \mu_{\text{prime-to-}p}(K)$$

Last time I wrote $R_K \cong \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$. group under addition

Claim: $1 + \pi R_K \cong \mu_{p\text{-power}}(K) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$

$$1 + \pi R_K \xrightarrow{\log} K \qquad \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

$\mathbb{Z}_p$-module

with bounded image. i.e. in $|z| \leq C$ for $C > 0$.

If $K = \mathbb{Q}_p$ then image lands in $p\mathbb{Z}_p$.

Not in general $\mathbb{Q}_2(\sqrt{2})$

$$\log(1+\sqrt{2}) = \sqrt{2} - \frac{(\sqrt{2})^2}{2} + \cdots$$
$$= \sqrt{2} - \boxed{1} + \cdots$$

Dominant term.

lands in $|z| \leq 1$.

Can show it has open image (can invert using exponential on $|z| < |p|^{\frac{1}{p-1}}$

$\Rightarrow$ image is a free $\mathbb{Z}_p$-module of rank $[K:\mathbb{Q}_p]$ = rank $R_K : \mathbb{Q}_p$.

(By structure of modules over a PID).

$\log : \quad 1+\pi\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$

Kernel $= \mu_{p\text{-power}}(K)$ $\leftarrow$ Need to check.

can use $(1+\pi\mathbb{Z}_p)^{p^N}$ for N large enough.

then $\subseteq 1+p^2R_K$

log is injective here.

Exact sequence $0 \to \mu_{p\text{-power}}(K) \to 1+\pi\mathbb{Z}_p \to \mathbb{Z}_p^{[K:\mathbb{Q}_p]} \to 0$ of $\mathbb{Z}_p$-modules

free.

So choose any splitting.

Get $\quad K^\times \cong \pi^{\mathbb{Z}} \times \mu_{\text{prime-to-}p}(K) \times \mu_{p\text{-power}}(K) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$

$\cong \pi^{\mathbb{Z}} \times \mu(K) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$

Key point in computation of K^{tr} for $K/\mathbb{Q}_p$ finite extension.

want to show $K^{tr} = K^{ur}(\pi^{1/N} \quad (N,p)=1)$

<u>Point:</u> If L/K^{ur} is a $\mathbb{Z}/N\mathbb{Z}$ extension w/ N coprime to p, then $L = K^{ur}(\pi^{1/N})$.

Use Kummer theory for K^{ur}:

$$(K^{ur})^\times \cong \pi^{\mathbb{Z}} \times \overline{\mathbb{F}}_p^\times \times 1+\pi R_{K^{ur}}$$

Compute and $$((K^{ur})^\times)^N = \pi^{N\mathbb{Z}} \times \overline{\mathbb{F}}_p^\times \times 1+\pi R_{K^{ur}}$$

Use Hensel's lemma to see everything has Nth root.

Note K^{ur} is not complete, BUT it's the union of finite (ur) extensions, each of which is complete.

$$(K^{ur})^\times / ((K^{ur})^\times)^N = \mathbb{Z}/N\mathbb{Z}$$

generated by π.

so my $\mathbb{Z}/N\mathbb{Z}$ extension is $K^{ur}(\pi^{1/N})$.

$K/\mathbb{Q}_p$ finite extension.

$$\begin{tikzcd}
\overline{K} = \overline{\mathbb{Q}}_p \arrow[d, dash, "P_K"'] \arrow[dd, bend left=60, dash, "I_K"] \arrow[dddd, bend left=80, dash, "G_K"] \\
K^{tr} \arrow[d, dash, "\hat{\mathbb{Z}}^{(p')}(1)"] \\
K^{ur} \arrow[d, dash, "G_{\overline{\mathbb{F}}_p/\kappa(K)}"'] \\
K
\end{tikzcd}$$

wild inertia group $\rightarrow$ P_K

G_K = abs. Galois group

I_K = abs. inertia group

P_K = abs. wild inertia group

$\uparrow$ pro-p group.

Have shown:

Theorem: $G_K/P_K = \mathrm{Gal}(K^{tr}/K) \cong G_{\overline{\mathbb{F}}_p/\kappa(K)} \ltimes \hat{\mathbb{Z}}^{(p')}(1)$

$\mathbb{Q}_p$

Question: What is the structure of P_K? Extension?

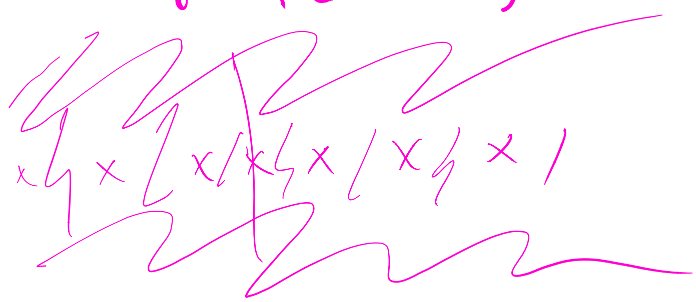
Theorem (Iwasawa On the Galois Groups of Local Fields)

The map $G_K \to G_K/P_K$ splits.

(so $G_K = P_K \rtimes G_K/P_K$)

and P_K is the pro-p completion of a countably generated free group.

E.g. $\pi_1(\mathbb{C}\setminus\mathbb{Z})$



Remark Nice algebraic presentation $\neq$ useful presentation.

Exercise Show that $(\mathbb{Q}_p^{tr})^\times / ((\mathbb{Q}_p^{tr})^\times)^p$

is an $\mathbb{F}_p$-vector space of countable dimension.

Simpler sub-exercise: Replace tr with ur.

$$\mathbb{Q}_p \subseteq \mathbb{Q}_p^{tr} \subseteq \overline{\mathbb{Q}_p}$$

Sketch:
$$\mathbb{Q}_p^{tr} = \bigcup_{(N,p)=1} \mathbb{Q}_p(\zeta_N, p^{1/N}).$$

Field of $X^N - p$

$\rightsquigarrow K_N =$ splitting field ... in $\overline{\mathbb{Q}_p}$.

<u>Claim:</u> $(\mathbb{Q}_p^{tr})^\times / ((\mathbb{Q}_p^{tr})^\times)^p$

$= \operatorname{colim} K_N^\times/(K_N^\times)^p$. $\Leftarrow$ easy.

AND $(1+\pi \mathcal{O}_{K_N}) / (1+\pi \mathcal{O}_{K_N})^p \hookrightarrow (\mathbb{Q}_p^{tr})^\times / ((\mathbb{Q}_p^{tr})^\times)^p$

(Otherwise get a wild subextension)

$\pi = p^{1/N}$

$$K_N^\times \cong \pi^{\mathbb{Z}} \times k(\mathcal{O}_{K_N})^\times \times (1+\pi \mathcal{O}_{K_N})$$

$$\downarrow \quad \mu_{p\text{-power}}(K_N) \times \mathbb{Z}_p^{[K_N:\mathbb{Q}_p]}$$

$$\wr$$

$$\mathbb{Z}/p\mathbb{Z} \times 1 \times \left\{ \begin{matrix} 1 \\ \mathbb{Z}/p\mathbb{Z} \end{matrix} \right\} \times \mathbb{F}_p^{[K_N:\mathbb{Q}_p]}.$$

$$\begin{array}{cccccc}
 & (\mathbb{Q}_p^{tr})^\times & \xrightarrow{p\text{-power}} & (\mathbb{Q}_p^{tr})^\times & \to & (\mathbb{Q}_p^{tr})^\times/((\mathbb{Q}_p^{tr})^\times)^p \to 1 \\
 & \| & & \| & & \\
\operatorname{colim} & (K_N^\times)_N & \xrightarrow{p\text{-power}} & (K_N^\times)_N & \to & (K_N^\times/(K_N^\times)^p)_N \to 1
\end{array}$$