<u>Last time:</u> Want to show maximal tamely ramified extension

$$K^{tr} \subseteq \overline{K} \leftarrow \text{sep. closure of } K$$

is "what we think it is" $= K^{ur}(\pi^{1/n} \quad (n,p)=1)$

uniformizer — residue characteristic

Recall (if this is true) $G_{K^{tr}/K} = G_{\overline{k(K)}/k(K)} \ltimes \hat{\mathbb{Z}}(1) \leftarrow$ residue char 0

$\hat{\mathbb{Z}}^{(p)}(1) \leftarrow$ residue char p.

$\varprojlim_N \mu_N(\overline{k(K)})$.

(Recall $K^{tr} = \overline{K}$ if equicharacteristic zero

otherwise $\underset{\overset{\shortparallel}{P_1}}{Gal(\overline{K}/K^{tr})}$ is a pro-p group).

To show this: use Kummer theory

↳ If L/K finite Galois tamely ramified

need to show $L \subseteq K^{tr}$.

<u>Exercise:</u> $\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^\ell$ for $\ell \neq p$ prime

$\ell = p$

$p=3 \quad \ell=2$

$$\mathbb{Q}_3^\times \cong 3^{\mathbb{Z}} \times \mathbb{Z}_3^\times$$
$$\cong 3^{\mathbb{Z}} \times \mathbb{F}_3^\times \times 1+3\mathbb{Z}_3$$

multiplicative lifts

2nd roots of units

$= \pm 1$.

$\log: 1+3\mathbb{Z}_3 \xrightarrow{\sim} 3\mathbb{Z}_3 \cong \mathbb{Z}_3$

exp

$\exp(u) = \sum^{\infty} \frac{u^n}{n!}$

$n=0$

have to check this is convergent $\sim |n!|_p$.

$$(\mathbb{Q}_3^\times)^2 \cong 3^{2\mathbb{Z}} \times (\mathbb{F}_3^\times)^2 \times (1+3\mathbb{Z}_3)^2$$

$$(\mathbb{F}_3^\times)^2 = \{1\}, \qquad (1+3\mathbb{Z}_3)^2 \cong 2\mathbb{Z}_3 = \mathbb{Z}_3$$

$$(\mathbb{Q}_3^\times)/(\mathbb{Q}_3^\times)^2 = \underline{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times 1}.$$

How many quadratic extensions of $\mathbb{Q}_3^\times$ are there?
Gives easily at most 3. (in fact exactly 3).

What's different for

$p=3 \quad l=3$

$$\mathbb{Q}_3^\times = 3^{\mathbb{Z}} \times \mathbb{F}_3^\times \times \mathbb{Z}_3$$

$\uparrow 1+3\mathbb{Z}_3$

$$(\mathbb{Q}_3^\times)^3 = 3^{3\mathbb{Z}} \times \mathbb{F}_3^\times \times 3\mathbb{Z}_3$$

$$(\mathbb{Q}_3^\times)/(\mathbb{Q}_3^\times)^3 = \mathbb{Z}/3\mathbb{Z} \times 1 \times \mathbb{Z}/3\mathbb{Z}.$$

All of these extensions are ramified.

$p=2 \quad l=2$

$$\mathbb{Q}_2^\times = 2^{\mathbb{Z}} \times \mathbb{F}_2^\times \times 1+2\mathbb{Z}_2$$

$$1+2\mathbb{Z}_2 \not\cong \mathbb{Z}_2 \quad !$$

-1 ; torsion free

$(-1)^2 = 1.$

$$1+2\mathbb{Z}_2 \cong \{\pm 1\} \times 1+4\mathbb{Z}_2$$

$$\{\pm 1\} \times \mathbb{Z}_2$$

$$1+2\mathbb{Z}_2 \to 1+4\mathbb{Z}_2$$

$$x \mapsto x^2$$

$$1+2\mathbb{Z}_2 \; . \; \log(x^2)/4 = \mathbb{Z}_2$$

Kernel $= \mu_2$.

$\mathbb{Q}_2^\times = 2^{\mathbb{Z}} \times 1 \times \mu_2 \times \mathbb{Z}_2$.

Exercise Let $L/\mathbb{Q}_p$ be a finite extension. Then: $L^\times \cong \pi^{\mathbb{Z}} \times \mu(L) \times R_L$

π uniformizer of L

$\mu(L)$: all roots of unity in L

R_L: $(\mathcal{O}_L)$ valuation ring as an additive group

Sketch: $R_L^\times \xrightarrow{x \mapsto x^N} 1 + p^2 R_L$

(Find such an N).

$R_L^\times \longrightarrow R_L$

$\ker = \mu(L)$. $x \longmapsto \log(x^N)$

use exp to find a splitting.

← Finding a splitting is a bit harder than I meant for this exercise. ~ will come back to this

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Let's prove if $K/\mathbb{Q}_p$ is tamely ramified Galois, then
$K \subseteq \mathbb{Q}_p^{ur}(p^{1/N} \forall (N,p)=1)$.
$\quad = \mathbb{Q}_p(\zeta_N, p^{1/N} \forall (N,p)=1)$.

$$\begin{array}{c}
\mathbb{Q}_p(\zeta_N, p^{1/N} \forall (N,p)=1) \\
\diagup \qquad \vdots\ ? \\
\mathbb{Q}_p^{ur} \qquad\qquad K \\
\diagdown \qquad \diagup e \\
K^I \quad \text{maximal unramified subfield.} \\
\text{unramified} \diagdown \\
\mathbb{Q}_p
\end{array}$$

$(e,p) = 1$

WLOG assume K contains all eth roots of unity.

tamely ramified.

$$\underset{\text{order } e}{\operatorname{Gal}(K/K^I)} \overset{\psi}{\hookrightarrow} \underset{\text{abelian}}{\mathcal{K}(K)^\times}$$

on char $\{e\} \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq H_3 \dots \trianglelefteq \operatorname{Gal}(K/K^I)$

$H_i/H_{i-1} \cong \mathbb{Z}/\ell\mathbb{Z}$ for a prime $\ell \mid e$

$\Rightarrow \ell \neq p$.

Reduced to assuming $e = \ell \neq p$ is prime.

So $\operatorname{Gal}(K/K^I) \cong \mathbb{Z}/\ell\mathbb{Z}$.

Thus K is obtained by adjoining an ℓth root of an element in K^I.

$$\underset{\overset{\parallel}{K_0^\times}}{(K^I)^\times} \cong p^{\mathbb{Z}} \times \mu(K_0) \times R_{K_0} \qquad \ell \in R_{K_0}^\times$$

$$(K_0^\times)/(K_0^\times)^\ell = \mathbb{Z}/\ell\mathbb{Z} \times \mu(K_0)/\mu(K_0)^\ell \times 1.$$

ℓth roots of roots of unity

This part corresponds to roots of p.

$$\mu(K_0) = \mu_{p\text{-power}}(K_0) \times \mu_{\text{prime-to-}p}(K_0)$$

$\mu_{p\text{-power}}(K_0) \xrightarrow[\ell]{\sim} \mu_{p\text{-power}}$ $\qquad$ $\mu_{\text{prime-to-}p}(K_0) \xrightarrow{\ell} \mu_{\text{prime-to-}p}(K_0)$

$$\mathbb{Z}/\ell\mathbb{Z} \times \mu_{\text{prime-to-}p}/(\mu_{\text{prime-to-}p})^\ell.$$

So, $K = K^I(\zeta^{1/\ell}(p^{1/n})^{1/\ell})$

ζ prime-to-p root of unity.

$$\subseteq K^{I}(\zeta^{1/\ell})(p^{1/n\ell}).$$

unramified

My argument was super messy.
Sorry.

Should have just immediately passed to $K \cdot \mathbb{Q}_p^{ur} / K$.

Added in discussion after class

$$(\mathbb{Q}_p^{ur})^\times \cong p^{\mathbb{Z}} \times \overline{\mathbb{F}}_p^\times \times 1 + p\mathbb{Z}_p^{ur}.$$

$$(\mathbb{Q}_p^{ur})^\times / ((\mathbb{Q}_p^{ur})^\times)^\ell \cong \mathbb{Z}/\ell\mathbb{Z} \times 1 \times 1$$

$x \mapsto x^\ell$ surjective on $\overline{\mathbb{F}}_p^\times$.