

Recall  $(K, |\cdot|)$  complete discretely valued.

If  $L/K$  finite Galois extension then:

$$\text{Ker} \rightarrow \text{Gal}(L/K) \rightarrow \text{Gal}(\mathcal{K}(L)/\mathcal{K}(K))$$

!!  $\mathcal{I}_{L/K}$  inertia subgroup.

= Galois group of  $L/L^{\text{ur}} \leftarrow$  maximal unramified subextension.

$\uparrow$  Extension is totally ramified.

Fix  $\pi$  a uniformizer  $R_L$  (= generator of  $\mathfrak{m}_L$ ).

$L = L^{\text{ur}}(\pi)$ . So  $\mathcal{I}_{L/K}$  completely determined by what it does to  $\pi$ . We know  $|\pi| = |\sigma(\pi)|$

for any  $\sigma \in \mathcal{I}_{L/K}$  (in fact in  $G_{L/K}$ ).

Defined a map  $\mathcal{I}_{L/K} \rightarrow \mathcal{K}(L)^{\times}$   
 $\sigma \mapsto \frac{\sigma(\pi)}{\pi} \bmod \pi$

Claimed this is a group homomorphism.

$\uparrow$  Uses that we've restricted to  $\mathcal{I}_{L/K}$ .

Define  $P_1 = \text{Ker } \mathcal{I}_{L/K} \rightarrow \mathcal{K}(L)^{\times}$

Exercise:  $P_1 = \left\{ \sigma \in \mathcal{I}_{L/K} \text{ (or } G_{L/K}) \text{ s.t. } \left| \frac{\sigma(\pi) - \pi}{\pi} \right| \leq |\pi| \right.$   
*(i.e.,  $\sigma(\pi) - \pi \in \mathfrak{m}_L^2 = (\pi)^2$  in  $R_L$ )*

$$\frac{\sigma(\pi)}{\pi} = 1 \bmod \pi \iff \sigma(\pi) = \pi + O(\pi^2).$$

Define  $P_r$   $r \geq 1$  by

$$P_r = \left\{ \sigma \in G_{L/K} \text{ s.t. } \left| \frac{\sigma(\pi) - \pi}{\pi} \right| \leq |\pi|^r \right. \\ \left. \left( \text{i.e. } \left( \frac{\sigma(\pi) - \pi}{\pi} \right) \in \frac{(\pi)^r}{\mathfrak{m}_L^r} \right) \right\}$$



Normal subgroups.

Have a map  $P_i / P_{i+1} \rightarrow m_L^i / m_L^{i+1}$

$\sigma \mapsto \frac{\sigma(\pi) - \pi}{\pi}$

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$$G_{L/K} \supseteq I_{L/K} \supseteq P_1 \supseteq P_2 \supseteq \dots$$

$$G_{\pi(L)/\pi(K)}$$

$$X(L)^X$$

$$m_1/m_2$$

$m_L^2/m_\tau^2$

511

511

511

$\chi(L)$

$$X(L)$$

812

(group law is addition)

Abelian quotients.

$$\chi(L) = \begin{array}{l} \text{Torsion free if } \text{char } K(L) = 0 \\ \text{or } p\text{-torsion if } \text{char } K(K) = 0 \end{array}$$

If  $X(L)$  is char 0 (e.g.  $K = \mathbb{C}((t))$ )

then  $P_i / \bigcap_{j \in I} P_j$  is trivial.

so  $P_1$  is trivial

Exercise:  $\bigcap_{i \in I} P_i = \{e\}$  (filtration is exhaustive).

Let  $\sigma \in \bigcap_{i=1}^{\infty} P_i$      $|\sigma(\pi) - \pi| \leq |\pi|^r$  for any  $r > 0$ .

$$\begin{aligned} \Rightarrow \quad & |\sigma(\pi) - \pi| = 0 \\ r \rightarrow \infty \quad & \sigma(\pi) = \pi. \end{aligned}$$

Summary  $\bigcap P_i = \{e\}$  so

$$G_{L/K} \supseteq I_{L/K} \supseteq P_1$$

$$\swarrow$$
  

$$G_{L/K} \supseteq I_{L/K} \supseteq P_1$$

$\uparrow$  p group if  $\text{char}(K) = p$   
 solvable  
 or trivial if  $\text{char}(K) = 0$ .

$$I_{L/K} / P_1 \leftrightarrow X(L)^{\times}$$

$$\searrow \downarrow \nearrow$$
  

$$N(X(L)) \leftarrow \text{Torsion elements in } X(L)^{\times}$$

Finite group

$\parallel$   
 group of roots of unity in  
 $X(L)^{\times}$ .

$$G_{L/K} / P_1$$

is simple.

(everything if residue field is  
 char 0)

$$I_{L/K} / P_1 \hookrightarrow N(X(L))$$

$\Rightarrow I_{L/K} / P_1$  has prime-to- $p$   
 order if  $X(L)$  is  
 char  $p$ .

$$(x^p - 1 = (x-1)^p \text{ in char } p).$$

$I_{L/K} / P_1$  is 1-torsion free  
 $P_1$  is  $p$ -group.

$\Leftarrow$  non-triviality of  $P_1$   
 $\leftrightarrow$

$p \mid e$  sufficient  
 degree of  $L/K$ .

$$(|G_{L/K}| = e).$$

$K \subseteq \bar{K} \hookrightarrow$  separable closure  
 $K^{ur} \hookrightarrow$  maximal unramified extension in  $\bar{K}$   
 $K^{tr} \hookrightarrow$  maximal tamely ramified extension in  $\bar{K}$ .  
 $\hookrightarrow (p_i \text{ is trivial for any subextension})$   
 $\Leftrightarrow p \nmid e$

$K^{ur} \hookrightarrow$  add roots to all Kth of irreducible polynomials in  $\mathcal{K}(K)[X]$ .

Note: this is not complete unless  $[\overline{\mathcal{K}(K)} : \mathcal{K}(K)] < \infty$ .

Fix  $\pi$  a uniformizer

$K^{ur}(\pi^{1/n} \mid \forall (n, p) = 1)$   $p = \text{char } \mathcal{K}(K)$ .

Theorem:  $K^{tr} \parallel$

(E.g.  $\mathbb{Q}(\zeta_{p^l})$   
 or  $\mathbb{Q}_p(\zeta_{p^l}, p^{1/n} \mid \forall (n, p) = 1)$

Need to show: Any tamely ramified ext<sup>n</sup> of  $K$  is contained in  $K^{ur}(\pi^{1/n} \mid \forall (n, p) = 1)$ .

Recall (from your part): Kummer theory:

IF  $F$  contains all  $N$ th roots of unity  
 and  $\text{char } F \nmid N$  then any  $\mathbb{Z}/N\mathbb{Z}$  extension  
 is obtained by adjoining an  $N$ th root.

Exercise: Compute  $\phi_p^x / (\phi_p^x)^x$   $l$  prime  $\neq$ .

(1) For  $l$  coprime to  $p$

(2) For  $l = p$ .