Recall: Last time - Fundamental inequality

$$[L:K] \le e(L/K)\, f(L/K)$$

K complete valued field — ramification (e), inertia (residue) (f)

$=$ if K is discretely value

Classification of complete discretely valued fields:

$k((t))$ if equichar. Finite (totally ramified) extension $W(k)[\frac{1}{p}]$.

(if mixed characteristic & k is perfect). (e.g. $\mathbb{Q}_p$ if $k = \mathbb{F}_p$).

Addition: A local field is a locally compact field $\implies$

$\mathbb{R}$, $\mathbb{C}$ or finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$ for some q. } Finite residue field, complete mixed characteristic.

(Arise when complete a global field ~ built from $\mathbb{Z}$ using finitely many algebraic steps), e.g. fin. extensions of $\mathbb{Q}$, or fin. extensions of $\mathbb{F}_q(t)$.

Today: Galois groups of complete discretely valued fields.

Fix $(K, |\cdot|)$ complete discretely valued.

$R = |\ | \le 1$ in K, $(\pi) = \mathfrak{m} \subseteq R$

π element of largest $|\ | < 1$.

v_π for π-adic valuation. $v_\pi(a) =$ power of π dividing a. $|a| = |\pi|^{v_\pi(a)}$.

$k = R/\mathfrak{m}$ $(= k(K))$.

First observation Suppose L/K is a finite Galois extension. induces an extension $k(L)/k(K)$ of residue fields.

$\mathrm{Gal}(L/K)$ preserves absolute value (because if $\sigma \in \mathrm{Gal}(L/K)$ $x \mapsto |\sigma(x)|$ is

preserves R_L & R_K

$$\mathfrak{m}_L \quad \& \quad \mathfrak{m}_K$$

$$\mathrm{Gal}(L/K) \circlearrowright R_L/\mathfrak{m}_L \supseteq R_K/\mathfrak{m}_K$$

$$\mathfrak{K}(L) \supseteq \mathfrak{K}(K)$$

$$\mathrm{Gal}(L/K) \to \mathrm{Aut}(\mathfrak{K}(L)/\mathfrak{K}(K)).$$

as absolute value extends absolute value on K

$\Rightarrow |\sigma(x)| = |x|$

Exercise Show $\mathfrak{K}(L)/\mathfrak{K}(K)$ is Galois and this map is surjective.

Sketch Let $a \in \mathfrak{K}(L)$ be a primitive element

$$\mathfrak{K}(L) = \mathfrak{K}(K)(a).$$

$$M_a(x) = x^f + c_{n-1}x^{f-1} \cdots + c_0.$$

lift the coefficients arbitrarily to $a \in L$

$$\tilde{M}(x) = x^f + \tilde{c}_{n-1}x^{f-1} + \cdots + \tilde{c}_0$$

$\tilde{c}_i \in R \qquad \tilde{c}_i = c_i \bmod (\pi).$ $\mathfrak{m}$

Irreducible (Gauss lemma) & L contains a root (lifting a via Hensel's lemma). (in fact R_L)

$\Rightarrow$ L contains all roots, they are Galois conjugate.

So all in R_L.

$\curvearrowright \mathrm{Gal}(L/K) \Rightarrow$ Reduce to all of the roots of $M_a(x)$.

$$\begin{array}{c} L \\ | \, e \\ K[x]/M_a(x) \\ | \, f \\ K \end{array}$$

$\leftarrow$ Galois extension.

action of $\mathrm{Gal}(L/K)$ on residue field factors through Galois group of this extension

Numerics $\Rightarrow$ Map is surjective.

Conclusion one part of the Galois group is described by Galois group of residue field extension (coefficients).

$$I_L \trianglelefteq \mathrm{Gal}(L/K) \twoheadrightarrow \mathrm{Gal}(k(L)/k(K))$$

I_L = kernel =: Inertia group

L^{I_L} ≈ maximal unramified extension (of K in L).

Exercise: If $L \supseteq M \supseteq K$ and M/K is unramified ($e=1$)

then $M \subseteq L^{I_L}$.

L^{I_L} and L have the same residue field

so L/L^{I_L} is totally ramified ($f=1$).

$$\begin{tikzcd}
L \arrow[d, dash, "I_L"'] & \text{totally ramified} \\
L^{ur} \arrow[d, dash, "\mathrm{Gal}(k(L)/k(K))"'] & = L^{I_L} \\
K & \text{unramified.}
\end{tikzcd}$$

Can split up any finite extension as unramified followed by totally ramified extension.

<u>Example</u> $\mathbb{C}((t^{1/3}))/\mathbb{R}((t))$

$$\begin{tikzcd}
 & \mathbb{C}((t^{1/3})) \arrow[d, dash, "\mathbb{Z}/3\mathbb{Z}"] \\
\mathbb{R}((t^{1/3})) \arrow[ur, dash, "\mathbb{Z}/2\mathbb{Z}"] \arrow[dr, dash] & \mathbb{C}((t)) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\
 & \mathbb{R}((t))
\end{tikzcd}$$

$$\mathbb{Z}/3\mathbb{Z} \to \mathrm{Gal}(\_/\_) \to \mathbb{Z}/2\mathbb{Z}$$

splits

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

Remark: Unramified extensions $\sim$ $K[x]/f(x)$ where $f(x) \in R[x]$, f mod m irreducible in $k[x]$.

Totally ramified extension can always use Eisenstein polynomials:

Exercise: If L/K is totally ramified then if $m_L = (\pi_L)$ $\left(\iff v_L(\pi_L) = \frac{1}{[L:K]} \iff |\pi_L| = |\pi_K|^{\frac{1}{[L:K]}} \right)$, [$L/K$ totally ramified]

then $L = K(\pi_L)$ and M_{π_L} is an Eisenstein polynomial

$n = [L:K]$

$$m_{\pi_L}(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$$

$a_i \in (\pi_K) = m \subseteq R_K$.

and $a_0 \in \pi_K$ but not in $(\pi_K)^2$.

i.e. $v_\pi(a_i) > 0$ and $v_\pi(a_0) = 1$,

i.e. $|a_i| < 1$ $\quad |a_0| = |\pi|$

Conversely: Any Eisenstein polynomial is irreducible & generates a totally ramified extension.

Claim: We "understand" $\mathrm{Gal}(k(L)/k(K))$

(Wrong; e.g. $K = \mathbb{Q}((t))$

t-adic valuation

But in practice true, e.g. $\mathbb{Q}_p$, $\mathbb{F}_p$)

Want to understand: $I_L = \mathrm{Gal}(L/L^{ur})$

Fix π_L generating $\mathfrak{m}_L \subseteq R_L$

$$L/L^{ur} \text{ is totally ramified}$$

$$L = L^{ur}(\pi_L).$$

E.g.:

$L = \mathbb{Q}_p(p^{1/2})$

$\mathbb{Q}_p = K$

$L^{ur} = K$

$\pi_K = p$

$\pi_L = p^{1/2}$

So action of I_L completely determined by what it does to π_L.

We know $|\sigma(\pi_L)| = |\pi_L|$

so we can look at

$$\begin{array}{ccccc} I_L & \longrightarrow & R_L^\times & \xrightarrow{\text{mod } \mathfrak{m}_L} & \mathfrak{K}(L)^\times \\ \sigma & \longmapsto & \dfrac{\sigma(\pi_L)}{\pi_L} & \longmapsto & \dfrac{\sigma(\pi_L)}{\pi_L} \text{ mod } \mathfrak{m}_L \end{array}$$

Lemma: The composition $I_L \to \mathfrak{K}(L)^\times$ is a homomorphism.

Example $\mathbb{Q}_p(p^{1/2}) / \mathbb{Q}_p$

$Gal = \langle \sigma \rangle$ s.t. $\sigma(p^{1/2}) = -p^{1/2}$.

$\pi = p^{1/2}$

$$\sigma \longmapsto \frac{\sigma(p^{1/2})}{p^{1/2}} \bmod p^{1/2}$$

$$\| $$

$$-1 \bmod p^{1/2} = -1 \in \mathbb{F}_p^\times$$

$$\langle \sigma \rangle \to \mathbb{F}_p^\times$$

$$\sigma \longmapsto -1.$$

Injection if $p \neq 2$.

Trivial if $p = 2$.

(This map captures tame ramification ← degree prime to char $k(K)$

wild ramification missed ← degree divisible char $k(K)$.)