Finally Completed proof that valuations extend uniquely to algebraic extensions of complete fields.

$(K, |\cdot|)$ complete valued field.

$$\left\{ \begin{array}{c} \text{Finite extensions} \\ \text{of } K \end{array} \right\} \xrightarrow{\text{Promotion!}} \left\{ \begin{array}{c} \text{Complete valued} \\ \text{finite extensions} \\ \text{of } K \end{array} \right\}$$

Today: Take advantage of this.

Notation All my fields are valued & complete, so $|\,|$ ← absolute value unambiguously.

If L/K finite we have valuation ring R_K ⤺ elements in K with $|\,| \leq 1$.

R_L ... in L with $|\,| \leq 1$.

Get $R_L \supseteq R_K$ ⤺ extension of fields promotes to an extension of rings.

$$\mathfrak{m}_L \supseteq \mathfrak{m}_K \;⤺\; |\,| < 1$$

$$\mathfrak{m}_K = R_K \cap \mathfrak{m}_L.$$

$\kappa(L) = R_L/\mathfrak{m}_L$ is an extension of $\kappa(K) = R_K/\mathfrak{m}_K$.

"Extension of valued fields induce extensions of coefficients"

Inertia index (degree) of L/K: $f(L/K) := [\kappa(L) : \kappa(K)]$.

We have $|K^\times| \leq |L^\times| \leq \mathbb{R}^\times_{>0}$

value groups

Ramification index (degree) of L/K: $e(L/K) := [|L^\times| : |K^\times|]$

⤺ Index of subgroup.

Example $\mathbb{R}((t)) \subseteq \mathbb{C}((t^{1/3}))$ ($|\,| =$ t-adic).

$\mathbb{R}[[t]] \subseteq \mathbb{C}[[t^{1/3}]]$ ← Valuation ring

$\mathbb{R} \quad \leq \quad \mathbb{C}$ ← Residue.

$f(\ldots) = 2.$

$|\mathbb{R}((t))^\times| = |t|^{\mathbb{Z}} \subseteq \mathbb{R}^\times_{>0}$

$|\mathbb{C}((t^{1/3}))^\times| = |t|^{\frac{1}{3}\mathbb{Z}} \subseteq \mathbb{R}^\times_{>0}$

$e(\ldots) = 3.$

Observation: $[\mathbb{C}((t^{1/3})) : \mathbb{R}((t))] = 6$ (add i then add $t^{1/3}$)

$= \underline{f \cdot e}.$

Theorem (Fundamental index inequality?) In general,

$$[L:K] \geq e(L/K) \cdot f(L/K).$$

Proof: Take representatives $l_1, \ldots, l_f \in R_L$ whose reduction to $\kappa(L)$ give a basis for $\kappa(L)/\kappa(K)$.

& representatives $m_1, \ldots, m_e \in L$

$|m_i|$ are coset representatives for $|L^\times|/|K^\times|$

Check that $l_i m_j$ $1 \leq i \leq f$, $1 \leq j \leq e$ are linearly independent$/K$. ∎

Note: It's not an equality in general.

If K is discretely valued then it is an equality.

In fact in this case

Exercise Show R_L is a free R_K-module of rank ef. (Implies $[L:K] = ef$)

because $L = R_L \otimes_{R_K} K$
$= R_L[\frac{1}{\pi}]$ for $\pi \in m_K$

$|K^\times|$ is a discrete subgroup of $\mathbb{R}^\times_{>0}$.

$\Rightarrow \exists\ \pi \in R_K$ with $|\pi|$ largest possible but <1.

$\Rightarrow |K^\times| = |\pi|^{\mathbb{Z}}$.

So $m_K = (\pi)$ is principal.

Observation: • Fundamental inequality $\Rightarrow$ L is discretely valued (property of discrete subgroups of $\mathbb{R}$).

• $m_L = (\eta)$ $\quad \eta^e$ generates $R_L \cdot m_K$

$\Rightarrow$ If we fix element $l_1, \ldots, l_f$ in R_L reducing to a $\mathfrak{K}(K)$-basis of $\mathfrak{K}(L)$

& Fix a set of rep's A for $\mathfrak{K}(K)$ in R_K.

we get a unique power series expansion

$$\sum_{k=0}^{\infty} \Big(\underline{\sum_{i=1}^{f} a_{i,k} l_i} \Big) \eta^k$$

$a_{i,k}$ are in A.

$\Rightarrow l_i \eta^k$ for $1 \le i \le f$ $\quad 0 \le k \le e-1$ are a basis.

By induction. Converges by completeness.

Theorem (Classification of (a lot) of complete discretely valued fields):
If $K, |\cdot|$ is complete discretely valued.

(1) If K is equicharacteristic then

$$K \cong \kappa(K)((t))$$

w/ t-adic absolute value up to equivalence.

(2) If K is mixed characteristic and $\kappa(K)$ is perfect of char p. then K is a finite extension of $W(\kappa(K))[\frac{1}{p}]$.

<u>Sketch:</u> In (1) ~ you need to find a copy of $\kappa(K)$ inside of K. Take the maximal subfield of R and show it has to be isomorphic to $\kappa(K)$.

In (2) ~ get a canonical map from Witt vectors to R ~ isomorphism on residue fields. Adjoin uniformizer in K.

<u>Note:</u> E.g. $\mathbb{F}_p((t))$ has still lots of interesting self-extensions

i.e. $L/\mathbb{F}_p((t))$ L abstractly iso to $\mathbb{F}_p((t))$.