$(K, |\cdot|)$ complete valued field L/K an algebraic extension $\Rightarrow$ unique extension of $|\,|$ to L. Moreover if L/K finite, $(L, |\cdot|)$ is complete.

- Showed assuming uniqueness and existence that for L/K finite
$$|\ell|_L = |N_{L/K}(\ell)|_K^{1/[L:K]}$$
- Most of the way through proving this formula defined an absolute value.

Last time.

Recall: We reduced to showing that for c s.t. $|N_{L/K}(c)|_K \leq 1$, then also $|N_{L/K}(1+c)| \leq 1$, with $L = K(c)$. To prove triangle inequality.

$$M_c(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \qquad a_0 = \pm N_{L/K}(c).$$

$$M_{1+c}(x) = M_c(x-1) = \underbrace{\dots}_{\text{constant term is } a_0 - a_1 + a_2 - a_3 \dots + (-1)^n} \quad // \pm N_{L/K}(c+1).$$

Suffices to show that $|a_i|_K \leq 1$ for $0 \leq i \leq n-1$.
(by ultrametric triangle inequality).

Know ① $M_c(x)$ is irreducible, monic
② $|a_0| \leq 1$.

Let i be s.t. $|a_i|_K$ is the largest among $|a_j|_K$, $1 \leq j \leq n-1$.

Suppose > 1.
$$a_i^{-1} M_c(x) = a_i^{-1}x^n + a_i^{-1}a_{n-1}x^{n-1} + \dots + x^i + a_i^{-1}a_{i-1}x^{i-1} \dots + a_i^{-1}a_0$$
$\in R[x]$ $\quad R$ = valuation ring $= \{\kappa \in K, |\kappa| \leq 1\}$.

Reduce mod $\mathfrak{m}$ ← maximal ideal in R $\rightsquigarrow$ get.

$$0 + \_x^{n-1} + \cdots \_x^{n-2} + \cdots + \_x, 0$$

$\hat{i}$ coeff of x^i is 1.

non-zero, & factors in $R/m[x]$.

Need to apply version of Hensel's Lemma that allows lifting factorizations.
$\rightsquigarrow$ get contradiction.

Homework $\rightsquigarrow$ look up this version, and complete the proof.

Complete <u>existence</u> of an absolute value on L extending $|\ |_K$.

<u>Uniqueness:</u>

Have L/K finite, $|\ |_1$ and $|\ |_2$ on L extending $|\ |$ on K.

<u>Lemma:</u> To show $|\ |_1 = |\ |_2$, suffices to show they define <u>equivalent</u> K-vector space norms

A norm here is $\|\cdot\|$ s.t.
$\|v_1 + v_2\| \le \max \|v_1\|, \|v_2\|$
$\|v\| = 0 \iff v = 0$
$\|kv\| = |k| \|v\|$.

$\exists\ a, b > 0$ s.t. $\forall\ l \in L$

$$* \quad a|l|_2 \le |l|_1 \le b|l|_2.$$

<u>Proof:</u> Let $l \in L$ apply $*$ to l^k for $k \in \mathbb{N}$.

$$a|l^k|_2 \le |l^k|_1 \le b|l^k|_2$$

$$a|l|_2^k \le |l|_1^k \le b|l|_2^k$$

take kth roots

$$a^{1/k}|l|_2 \le |l|_1 \le b^{1/k}|l|_2$$

$a^{1/k}, b^{1/k} \xrightarrow[k \to \infty]{} 1$ so squeeze theorem $\Rightarrow$

$$|\ell|_2 = |\ell|_1 .$$

Thus, only need to show:

Theorem: If V is a finite dimensional vector space over a complete valued field, then any two norms on V are equivalent and V is complete for any norm.

<u>High-powered proof:</u>

① Any 2 linear topologies on a f.d. v.s. over a completed valued field are equivalent.

② In particular, V is complete in any norm topology (because K^n w/product topology is complete)

③ $Id: (V, \| \ \|_1) \to (V, \| \ \|_2)$ is a continuous surjection $\Rightarrow$

(open mapping theorem). ⇝ norms are equivalent.

Image of unit ball contains smaller ball

standard proof of open mapping theorem applies.

Proving this ~ as proving Theorem.

<u>Remark:</u> If you restrict $(K, |\cdot|)$ ← discretely valued ← locally compact.

then there are easier proofs.

more obvious

Proof of Theorem:

$(K, |\ |)$ complete valued field.

Reduction: Suffices to show that any norm $\|\cdot\|$ on K^n is equivalent to $\|\cdot\|_\infty$ ← sup norm

$$\left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\|_\infty = \max_{i=1,\dots,n} |a_i|.$$

(K^n is obviously complete for $\|\cdot\|_\infty$).

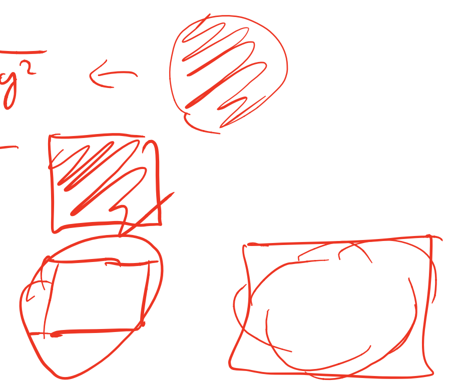
Picture

$K = \mathbb{R}$

$\mathbb{R}^2$, $\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_2 = \sqrt{x^2+y^2}$ ←

$\|\ \|_\infty$ ←

Need to show



Proof: By induction on n. (Looking at K^n).

Base case: $n=1$ Compare $\|\ \|$ w/ $\|\ \|_\infty$

$$\|[\lambda]\| = \|\lambda[1]\| = |\lambda|\,\|[1]\|$$
$$= \|[\lambda]\|_\infty \cdot \underline{\|[1]\|}$$

So in fact $\|\ \|$ is just a multiple of $\|\ \|_\infty$.

Inductive step: Easy: $\|\sum a_i e_i\|$ e_i std basis vectors

$$\leq \max \|a_i e_i\| = \max |a_i|\,\|e_i\|$$
$$\leq \max|a_i| \max\|e_i\|$$
$$= \underline{\max\|e_i\|} \cdot \|\textstyle\sum a_i e_i\|_\infty.$$

So $\|v\| \leq c\|v\|_\infty$.

Need to show $\|v\|_\infty \leq c\|v\|$.

Suppose not: Then, up to reordering the basis, I can find a sequence $V_i \in K^n$ s.t.

$$V_i = \begin{bmatrix} a_{i,1} \\ \vdots \\ a_{i,n} \end{bmatrix}$$

s.t. $a_{i,n} = 1$ for all i.

and $\|v_i\| \to 0$.

Sketch of how: for any c, can find v with $\|v_\infty\| \geq c\|v\|$.

Look at $w_i = V_i - e_n \in K^{n-1} = \begin{bmatrix} * \\ \vdots \\ * \\ 0 \end{bmatrix}$.

$\| \ \| |_{K^{n-1}}$ equivalent to $\| \ \|_\infty$ on K^{n-1}. Ind. hypothesis.

Show Cauchy for $\| \ \|$.

$$\|w_i - w_j\| = \|v_i - e_n - (v_j - e_n)\| = \|v_i - v_j\| \leq \max\|v_i\|, \|v_j\| \to 0$$

So (w_i) Cauchy in K^{n-1} for $\| \ \|$

by inductive hypothesis,

so $w_i \to w$ for $\| \ \|$ in K^{n-1}

in K^n, $w_i \to -e_n$ for $\| \ \|$

(because $v_i \to 0$).

$-e_n \notin K^{n-1} \leadsto$ contradiction.