Theorem: If $(K, |\cdot|)$ is a complete valued field and L/K is an algebraic extension, then there is a unique extension of $|\cdot|$ to L.

Example: $(K, |\cdot|) = (\mathbb{Q}_p, |\ |_p)$ and fix $\overline{\mathbb{Q}}_p$ an algebraic closure then it makes sense to write $|a|_p$, $a \in \overline{\mathbb{Q}}_p$.

Enhanced Theorem: If $(K, |\cdot|)$ is a complete valued field and L/K is a finite extension, then
$$\ell \mapsto |N_{L/K}(\ell)|^{1/[L:K]}$$
is the unique extension of $|\ |$ to L.

Finite extensions $\Rightarrow$ Algebraic extensions.

If L/K is algebraic and $\ell \in L$.
then define $|\ell|_L = |\ell|_{K(\ell)}$

↑ unique extension of $|\ |$ to $K(\ell)$.

E.g. If $\ell_1, \ell_2 \in L$

then
$$\begin{aligned}
|\ell_1 + \ell_2| &= |\ell_1 + \ell_2|_{K(\ell_1+\ell_2)} \\
&= |\ell_1 + \ell_2|_{K(\ell_1, \ell_2)} \\
&\leq |\ell_1|_{K(\ell_1,\ell_2)} + |\ell_2|_{K(\ell_1,\ell_2)} \\
&\leq |\ell_1|_{K(\ell_1)} + |\ell_2|_{K(\ell_2)}.
\end{aligned}$$

Norms: If L/K is a fin. ext. of Fields

$N_{L/K}(\ell) = \det$ of mult. by ℓ. mult. by ℓ is a K-linear map on the K-vector space ✓.

Exercise: (1) If $k \in K$ then $N_{L/K}(k) = k^{[L:K]}$

(2) If $\ell \in L$ and $M = K(\ell)$

then $N_{M/K}(\ell) = \pm$ constant coeff of the minimal polynomial of ℓ.

✓

(1) Take any basis $e_1, \ldots, e_n$

the matrix of $\cdot k$ is

$$\begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix}$$

✓

(2) $\ell \in L$ $\quad M_\ell(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$ $\quad M = K(\ell)$.

$a_i \in K$ $\quad m = [M : K]$

Take basis $1, \ell, \ell^2, \ldots, \ell^{m-1}$

mult. by ℓ in this basis

$$\begin{pmatrix} 0 & & & 0 & \boxed{-a_0} \\ \boxed{1} & 0 & & & \vdots \\ & \boxed{1} & 0 & & \vdots \\ & & \ddots & 0 & -a_{m-2} \\ 0 & & & \boxed{1} & -a_{m-1} \end{pmatrix}$$

$\ell^m = \ell \cdot \ell^{m-1}$

$-a_{m-1}\ell^{m-1} - a_{m-2}\ell^{m-2} + \cdots - a_0$

plug in $M_\ell(\ell) = 0$.

$\det = (-1)^m a_0$

Assuming there is a unique extension, we'll deduce the Formula:

$L/K \qquad |\ |_K \qquad |\ |_L$ = unique extension.

<u>Observation</u> If $\sigma \in Aut(L/K)$ then $\ell \mapsto |\sigma(\ell)|_L$ is an absolute value on L extending the absolute value on K, so it is equal to $|\ |_L$

i.e. $|\ell|_L = |\sigma(\ell)|_L$

($Aut(L/K)$ automatically preserves $|\ |$ so acts by isometries);

Let's assume L/K is separable (skip inseparable case).

Can take a Galois extension N/K containing L.

$\ell \in L$

$$|N_{L/K}(\ell)|_K = \left| \begin{matrix}\text{0th coefficient} \\ \text{of } m_\ell(x)\end{matrix} \right|_K^{[L:K(\ell)]}$$

$$= \left| \begin{matrix}\text{product of the} \\ \text{roots of } m_\ell(x) \\ \text{in } N\end{matrix} \right|_K^{[L:K(\ell)]}$$

$\alpha_1 = \ell$

$$= |\alpha_1 \alpha_2 \cdots \alpha_m|_K^{[L:K(\ell)]}.$$

$$= |\alpha_1 \cdots \alpha_m|_N^{[L:K(\ell)]}$$

$$= \left(\prod_{i=1}^{m} |\alpha_i|_N\right)^{[L:K(\ell)]}$$

$$= \prod_{i=1}^{n} |\sigma_i(\alpha_1)|_N^{[L:K(\ell)]}$$

$\sigma_i \in \mathrm{Gal}(N/K)$ sending α_1 to α_i

$$= \prod_{i=1}^{n} |\alpha_1|_N^{[L:K(\ell)]}$$

$$= |\alpha_1|_N^{[L:K]}$$

$$= |\ell|_N^{[L:K]}$$

$$= |\ell|_L^{[L:K]}$$

$$|N_{L/K}(\ell)|^{1/[L:K]} = |\ell|_L .$$

Left to do

- Show that $|\ell|_L = |N_{L/K}(\ell)|^{1/[L:K]}$ defines a absolute value on L.
- Show that if $|\ |_1$ and $|\ |_2$ both extend then they are the same.

Check this first.

From exercise have

$$|N_{L/K}(x)|_K^{1/[L:K]} = |x|_K$$

So if it is an absolute value it extends $|\ |_K$.

$$|N_{L/K}(l_1 l_2)|_K^{1/[L:K]} = |N_{L/K}(l_1) N_{L/K}(l_2)|_K^{1/[L:K]}$$

$$= |N_{L/K}(l_1)|_K^{1/[L:K]} |N_{L/K}(l_2)|_K^{1/[L:K]}$$

$$|N_{L/K}(l_1)|_K^{1/[L:K]} \neq 0 \quad \text{for } l_1 \neq 0$$

because mult. by l_1 is invertible.

Remains to check strong triangle inequality.

$$|N_{L/K}(l_1 + l_2)|_K^{1/[L:K]} \leq \max \ldots l_1 \ldots, \ldots l_2 \ldots$$

$$\Updownarrow$$

$$|N_{L/K}(l_1 + l_2)|_K \leq \max |N_{L/K}(l_1)|_K, |N_{L/K}(l_2)|_K$$

Suppose $|N_{L/K}(l_1)| = \max |N_{L/K}(l_1)|_K, |N_{L/K}(l_2)|_K$

multiply by $|N_{L/K}(l_1)^{-1}| = |N_{L/K}(l_1^{-1})|$

$$\Updownarrow$$

$$|N_{L/K}(1+\ell)| \leq 1.$$

For all ℓ w/ $|N_{L/K}(\ell)| \leq 1.$

$$\Updownarrow$$

ℓ w/ M_ℓ has constant coefficient with $|\ | \leq 1.$

$\Rightarrow M_{1+\ell}$ has constant coefficient with $|\ | \leq 1.$

Exercise Try to show this.

$$M_{1+\ell}(x) = M_\ell(x-1) \qquad M_\ell(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$$

constant coefficient is

$a_0 - a_1 + a_2 \cdots (-1)^m a_m.$

Know $|a_0| \leq 1$ and $M_\ell(x)$ is irreducible and monic.

Can show $\Rightarrow |a_k| \leq 1 \quad 0 \leq k \leq m.$

Version of Hensel's lemma.

Next time