## Last time

Examples of valued fields

Complete / not complete — Discrete / not discrete

$(\mathbb{Q}_p, |\ |_p)$ $\quad$ $(\mathbb{Q}, |\ |_p)$

$(\mathbb{C}((t)), |\ |_t)$ $\quad$ $(\mathbb{C}(t), |\ |_t)$

$(\mathbb{Q}_p(p^{1/\infty}), |\ |_p)$

$(\mathbb{Q}_p(p^{1/\infty})^\wedge, |\ |_1)$

$(\mathbb{C}((t^{1/\infty})), |\ |_t)$

perfect / imperfect residue field

$(\mathbb{Q}_p, |\ |_p)$ $\quad$ $\mathbb{F}_p$

$(\mathbb{C}((t)), |\ |_t)$ $\quad$ $\mathbb{C}$

$(\mathbb{Q}_p(p^{1/\infty})^\wedge, |\ |_p)$

$(\mathbb{Q}_p(p^{1/\infty}), |\ |_p)$

$(\mathbb{Q}_p(p^{1/N}), |\ |_p)$

$\mathbb{F}_p$

In class I forgot to recall

$\sum_{n \in \mathbb{Z}} a_n t^n \quad a_n \in \mathbb{Q}_p$, $|a_n|$ bounded, $a_n \to 0$ as $n \to -\infty$

as example with imperfect residue field.

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## Extensions of valued fields $(L, |\ |_L)/(K, |\ |_K)$

means field extension & absolute values agree on subfield.

means $K \hookrightarrow L$ and $(|\ |_L)\big|_K = |\ |_K$.

## Example $\mathbb{Q}(i)$.

Facts: 1. $\mathbb{Q}(i) = \mathrm{Frac}(\mathbb{Z}[i])$

$\mathbb{Z}[i]$ is a unique factorization domain.

2. If $\pi \in \mathbb{Z}[i]$ is irreducible can define the $\pi$-adic valuation / abs value $|\ |_\pi$

power of $\pi$ dividing $f$

$v_\pi(f) = n$ s.t.

$f = \pi^n \frac{s}{t}$

$s, t \in \mathbb{Z}[i]$

$\pi \nmid s, \pi \nmid t$.

<u>Question:</u> How can I extend the following absolute values from $\mathbb{Q}$ to $\mathbb{Q}(i)$.

$| \, |_\infty$ $\quad$ $| \, |_2$ $\quad$ $| \, |_3$ $\quad$ $| \, |_5$

$2$ Archimedean abs. value

$2 = (1+i)(1-i)$

$v_{1+i} = v_{1-i}$

Say $| \, |$ is an extension (finite)

$\mathbb{Q}(i)^\wedge / \mathbb{Q}^\wedge$ = extension of $\mathbb{R}$

contains $i$

so isomorphic to $\mathbb{C}$

$\mathbb{Q}(i) \to \mathbb{C}$ $\quad$ $| \, |_\mathbb{C}$

$i \longmapsto i$

$i \longmapsto -i$.

$| \, |_\mathbb{C} \big|_{\mathbb{Q}(i)}$ for either embedding extends $| \, |_\infty$

these two extensions are the same.

(There is a unique extension of $| \, |_\infty$ on $\mathbb{R}$ to $| \, |$ on $\mathbb{C}$).

$v_{1+i}(2) = \underline{2}$

$(1-i) = -i(1+i)$

$2 = (-i)(1+i)^2$

unit $\quad$ irreducible.

this valuation does not extend $v_2$ on $\mathbb{Z}$.

$\frac{1}{2} v_{1+i}$ extends $v_2$

(equivalence class of $v_{1+i}$ extends equivalence class of $v_2$).

$|\ |_3$

?

Need to factor 3 in

$$\mathbb{Z}[i] = \mathbb{Z}[x]/x^2+1.$$

Claim 3 is irreducible.

(need to check $\mathbb{Z}[i]/3$ integral domain)

$$= \mathbb{Z}/3\mathbb{Z}[x]/x^2+1$$

$$\mathbb{F}_3[x]/x^2+1$$

$x^2+1$ is irreducible, so this field.

So $|\ |_3$ on $\mathbb{Q}(i)$ makes sense.

(and is the unique extension).

$|\ |_5$ $\qquad 5 = (2+i)(2-i).$

Check these don't differ by $\pm 1, \pm i$ = units in $\mathbb{Z}[i]$.

So non-equivalent irreducibles.

$$v_{2+i}(5) = 1 \qquad v_{2+i}(2-i) = 0$$

$$v_{2-i}(5) = 1 \qquad v_{2-i}(2-i) = 1$$

← so not same extension.

Two distinct extensions
To factor 5:

$$\mathbb{Z}[i]/5 = \mathbb{F}_5[x]/x^2+1$$

$$\begin{tikzcd}
\mathbb{F}_5[x]/x^2+1 \arrow[d, phantom, "\shortparallel"] & x \arrow[d, bend right, mapsto] \\
\mathbb{F}_5 \times \mathbb{F}_5 & (3,2).
\end{tikzcd}$$

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$$\begin{tikzcd}
\mathbb{Q}(i)^{\wedge \leftarrow |\,|_{\mathbb{C}}} \arrow[r, phantom, "="] & \mathbb{C} \\
\mathbb{Q}^{\wedge} \arrow[u] \arrow[r, phantom, "="] & \mathbb{R} \arrow[u]
\end{tikzcd}$$

$\uparrow$ $|\,|_\infty$

$$\begin{tikzcd}
\mathbb{Q}(i)^{\wedge \leftarrow |\,|_{1+i}} \arrow[r, phantom, "="] & K \arrow[d, "\text{degree 2 extension}"] \\
\mathbb{Q}^{\wedge} \arrow[u] \arrow[r, phantom, "="] & \mathbb{Q}_2
\end{tikzcd}$$

$\uparrow$ 2-adic

$$\begin{tikzcd}
\mathbb{Q}(i)^{\wedge \leftarrow |\,|_3} \arrow[r, phantom, "="] & W(\mathbb{F}_9)[\frac{1}{p}] \\
\mathbb{Q}^{\wedge} \leftarrow |\,|_3 \arrow[u] \arrow[r, phantom, "="] & \mathbb{Q}_3 = W(\mathbb{F}_3)[\frac{1}{p}] \arrow[u]
\end{tikzcd}$$

$$\mathbb{Q}(i)^{\wedge} \leftarrow |\ |_{2+i} = \mathbb{Q}_5$$
$$\|$$
$$\mathbb{Q}^{\wedge} \leftarrow |\ |_5 = \mathbb{Q}_5$$

$$\mathbb{Q}(i)^{\wedge} \leftarrow |\ |_{2-i} = \mathbb{Q}_5$$
$$\|$$
$$\mathbb{Q}^{\wedge} \leftarrow |\ |_5 = \mathbb{Q}_5$$

The point here is $\mathbb{Q}_5$ contains two roots of $-1$ get 2 distinct embeddings of $\mathbb{Q}(i)$ into it.

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Observation: Extensions of absolute values are not unique. in general ~ i.e. if $K, |\cdot|_K$ and $L/K$ then there may be multiple $|\cdot|$ on $L$ extending $|\ |_K$. We'll see: if $(K, |\cdot|_K)$ is complete & $L/K$ is algebraic then there is a unique extension of $|\ |_K$ to $L$.

Consequence

The term $p$-adic propagates accurately to all algebraic extensions of $\mathbb{Q}_p$.

Exercise:

$\mathbb{C} \sqcup \{\infty\}$ = Riemann sphere = $\mathbb{P}^1(\mathbb{C})$

$\mathbb{C}(t)$ for every point $p$ in $\mathbb{P}^1(\mathbb{C})$

I get a valuation

= order of vanishing at $p$.

If $z \neq \infty$ then

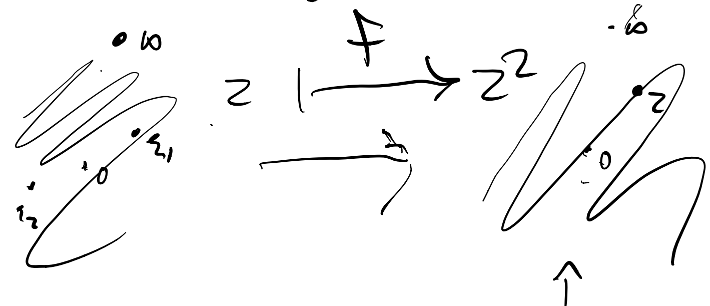
$$\underline{v_z = v_{(t-z)}}$$

$t-z \leftarrow$ irreducible in $\mathbb{C}[t]$

$$\begin{array}{ccc} \mathbb{C}(s) & & s^2 \\ \uparrow & & \uparrow \\ \mathbb{C}(t) & & t \end{array}$$

degree 2 field extension

$\leftarrow$ pullback of meromorphic functions via $f$.




variable $s$ variable $t$

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Q: How do these valuations extend to $\mathbb{C}(s)$?

$z \neq 0, \infty$ $\quad v_z \to v_{z_1}, \ v_{z_2}$ $\quad z_1^2 = z_2^2 = z$

$z = 0$ $\quad v_0 \to \frac{1}{2} v_0$

$z = \infty$ $\quad v_\infty \to \frac{1}{2} v_\infty$

} Ramification.

**Theorem:** If $(K, |\cdot|)$ is a complete valued field and $L/K$ is an algebraic extension then there is a unique extension of $|\cdot|$ to $L$. Moreover, if $L/K$ is finite then $(L, |\cdot|)$ is complete.

If $L/K$ is finite
The extension of $|\cdot|$ is given by

$$|\ell| := |N_{L/K}(\ell)|^{1/[L:K]}$$

this the determinant of mult. by $\ell$ on the $K$ vector space $L$.