Non-archimedean valued fields:

A pair $(K, |\cdot|)$ s.t. K is a field, $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is an ultrametric absolute value.

(Reminder $|\cdot| \longmapsto \underset{\text{valuation}}{\log_c |\cdot|}$ for any $c>1$

$\underset{\text{for any } c>1}{c^v} \longleftarrow v : K \to \mathbb{R} \sqcup \{-\infty\}$)

2 valuations are equivalent if $v_1 = c v_2$ for some $c \in \mathbb{R}_{>0}$.

Valuation $\log_c |\cdot|$ is a rank 1 valuation if you look more generally.

The valuation ring R for $(K, |\cdot|)$ is $\{z \in K \mid |z| \leq 1\}$ or $v(z) \geq 0$

<u>Observation</u>: R is a local ring with maximal idea

$$\mathfrak{m} = |z| < 1. \quad (v(z) > 0).$$

<u>Residue field</u> $\mathcal{K} := R/\mathfrak{m}$

<u>Example</u> $(\mathbb{Q}_p, |\cdot|_p)$

$$R = \mathbb{Z}_p \qquad \mathfrak{m} = p\mathbb{Z}_p$$

$$\mathcal{K} = \mathbb{Z}_p / p\mathbb{Z}_p = \mathbb{F}_p.$$

Non-archimedean valued fields

	Field	Abs. value (valuation)	Valuation ring	Residue field
(-1)	$\mathbb{Q}_p$	$\lvert\ \rvert_p$	$\mathbb{Z}_p$	$\mathbb{F}_p$
(0)	$\mathbb{Q}_3(\zeta_5) = W(\mathbb{F}_{81})[\frac{1}{3}]$	$\lvert\ \rvert_p$ Anything $= p^n a$, $a = [\xi_1] + p[\xi_2] + \dots$, $\xi_i \in \mathbb{F}_{81}$, $\xi_1 \neq 0$; order of vanishing at $p=0$	$W(\mathbb{F}_{81})$	$\mathbb{F}_{81}$

$$\mathbb{Q}_3(\zeta_5) \not\cong \mathbb{Q}_3[x]/\boxed{x^5-1}$$

not minimal polynomial.

$$x^4+x^3+x^2+x+1.$$

Irreducible by arguing with residue field.

(what is $\mathbb{F}_3(\zeta_5)$?

$$\mathbb{F}_3(\zeta_5) = \mathbb{F}_{81} = \mathbb{F}_{3^4}$$

$$\mathrm{Gal}(\mathbb{F}_3(\zeta_5)/\mathbb{F}_3) = \langle \mathrm{Frob} \rangle$$

$$\zeta_5 \xrightarrow{F} \zeta_5^3 \xrightarrow{F} \zeta_5^9 \xrightarrow{F} \zeta_5^{27} \to \zeta_5^{81} = \zeta_5$$

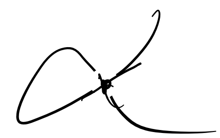
because 81 is 1 mod 5.

$\mathbb{Q}_3(\zeta_5) \cong W(\mathbb{F}_{81})[\frac{1}{3}]$.

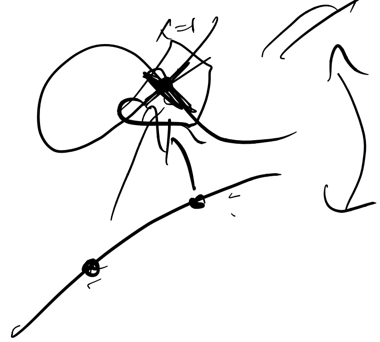
① $W(\overline{\mathbb{F}_p})[\frac{1}{p}]$	$\lvert \ \rvert_p$	$W(\overline{\mathbb{F}_p})$	$\overline{\mathbb{F}_p}$.
② $W(k)[\frac{1}{p}]$ for any k perfect field.	$\lvert \ \rvert_p$	$W(k)$	k
③ $\mathbb{F}_p(t)$	$f(t)$ irreducible in $\mathbb{F}_p[t]$ $\lvert \cdot \rvert_{f(t)}$ $h \in \mathbb{F}_p(t) = f(t)^n \frac{r(t)}{s(t)}$ where $f(t) \nmid r(t)$ $f(t) \nmid s(t)$	$\mathbb{F}_p[t]_{(f(t))}$	$\mathbb{F}_p[t]/f(t)$.
④ R a dvr = reduced local PID $= (\pi)$ $\operatorname{Frac} R$	π-adic $\lvert \ \rvert_\pi$	R	R/π
R any integral domain $\mathfrak{p}$ prime such that $R_\mathfrak{p}$ is a PID $\operatorname{Frac} R$	$\lvert \ \rvert_\mathfrak{p}$ $\operatorname{Frac} R \cong \operatorname{Frac} R_\mathfrak{p}$	$R_\mathfrak{p}$	$R_\mathfrak{p}/\mathfrak{p} = k(\mathfrak{p})$.

(5)

Spec R geometric,
then this is
order of vanishing / pole
of meromorphic function
along a codimension 1
smooth subvariety.



Nonexample



$k[x,y]/y^2 = x(x-1)^2$

$\mathfrak{m} = (x,y)$.

Have same
fraction field

Properties of valued fields

(1) Complete or not complete? Complete ~ e.g. $W(k)[\frac{1}{p}]$ $|\ |_p$

Not complete ~ e.g. $\mathbb{F}_p(t)$ $|\ |_t$

(2) Mixed characteristic or
equicharacteristic?

char $K \neq$ char k ; char $K =$ char k

(3) Discretely valued or not

$v(K^\times)$ is a discrete subgroup of $\mathbb{R}$.

(4) Perfect vs. imperfect residue field

Exercise Write down $(K, |\cdot|)$ that is complete, mixed characteristic and.

① Not discretely valued
② Imperfect residue field.

} 2 different examples.

(1) $\mathbb{Q}_p(p^{1/\infty}) = \operatorname{colim}_n \mathbb{Q}_p(p^{1/n})$

$\nearrow$

check that $|\ |_p$ extends by $|p^{1/n}| = p^{-1/n}$.

value group $= \mathbb{Q} \subseteq \mathbb{R}$.

But not complete.

$\mathbb{Q}_p(p^{1/p^\infty})^\wedge \leftarrow$ completion.

$\sum_{n=1}^{\infty} p^{n+\frac{1}{n}}$ converges but not to an element of $\mathbb{Q}_p(p^{1/\infty})$.

(2) $K =$ the ring of Laurent series w/coeff. in $\mathbb{Q}_p$ in variable t

$\sum_{n=-\infty}^{\infty} a_n t^n$ $\quad |a_n| \to 0$ as $n \to -\infty$ $\quad |a_n|$ is bounded.

Exercise: This is a field.

$$|f| = \sup |a_n| = \max |a_n|$$

Valuation ring $\sum_{n=-\infty}^{\infty} a_n t^n$

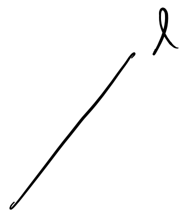
$a_n \in \mathbb{Z}_p$ and $|a_n| \to 0$ as $n \to -\infty$.

Residue field $= \mathbb{F}_p((t))$.

$\hookleftarrow$ Not perfect.

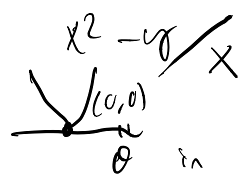
$\mathbb{A}^2$

$\mathbb{C}(x,y)$



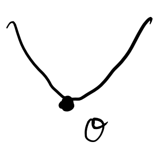
$\rightsquigarrow$ $v_l(f)$ = order of pole or vanishing of f along l.

Example of rank 2 valuation from discussion after class.



in $\mathbb{A}^2$

$v_O(f)$ = "order of pole or vanishing at $(0,0)$"



C = e.g. $x^2 - y$.

$f \in \mathbb{C}(x,y)$

$f = (x^2-y)^{v_C(f)}\, g$. g has no pole or zero along C.

$g|_C$ is a rational function on C.

$V_0(g)$.

Rank 2 - Valuation $\quad V_{C,0}(f) = \left(v_C(f), v_0\left(\frac{f}{(x^2-y)^{v_C(f)}}\right)\right)$

$$\rotatebox{90}{$\in$}$$

$$\mathbb{Z} \times \boxed{\mathbb{Z}},$$

Dictionary order

$0, 1$

$V(f) = (0, 1)$

$v(f) > V(1) = (0,0)$.

but $\quad V(f^n) = (0, n)$

doesn't become arbitrarily small

$f^n \not\to 0$.