Last time: $\mathbb{R}((t))$ computed the Galois group of its algebraic closure.

Today: Analog for $\mathbb{Q}_p$

1. Infinite galois theory:

(Source: Szamuely - Galois groups & Fundamental groups).
Ch 3(?) - Riemann-Mac..

Def'n: An extension of fields L/K is *Galois* if it is

1) algebraic
2) normal ⇝ Every polynomial w/ coeffs in K w/ a root in L splits in L
3) separable ⇝ If f irreducible in K has a root in L then f has no multiple roots.

$$\mathbb{F}_p(t) \hookrightarrow \mathbb{F}_p(t^{1/p}) \cong \mathbb{F}_p(t)[x]/x^p - t$$

$$x^p - t = (x - t^{1/p})^p$$

Splitting field for $x^p - t$

But has no non-trivial automorphisms $(\mathbb{F}_p(t^{1/p}) / \mathbb{F}_p(t))$

E.g. $\mathbb{F}_p((t)) \hookrightarrow \mathbb{F}_p((t^{1/p}))$

Observation If L/K is algebraic and $\alpha \in L$ then $K(\alpha)$
smallest subfield of L containing

$K \subseteq \alpha$

is a finite extension of K

$$K(\alpha) \cong K[x]/_{M_\alpha(x)}$$

$\alpha \longleftarrow x$ minimal polynomial

More generally if $\alpha_1, \dots, \alpha_n \subseteq L$

then $K(\alpha_1, \dots, \alpha_n)$ is an algebraic extension.

$\Rightarrow$ If L/K is a Galois extension

then $L = \bigcup_{L \supseteq M \supseteq K} M$ $\quad M/K$ finite Galois.

<u>Last time:</u> $\mathrm{Gal}(L/K) := \mathrm{Aut}(L/K)$

profinite group

$$= \lim_{\substack{L \supseteq M \supseteq K \\ M/K \text{ Galois.}}} \mathrm{Gal}(M/K)$$

topological group (limit topology for discrete topology on $\mathrm{Gal}(M/K)$).

= Coarsest topology s.t.

$\mathrm{Gal}(L/K) \times L \to L$ $\quad$ L has discrete topology.

$(\sigma, \ell) \mapsto \sigma(\ell)$

is continuous

Fundamental theorem of Galois theory:

$\mathrm{Gal}(L/K) \supseteq H \longmapsto L^H$ $\leftarrow$ fixed field

$= \{\ell \in L \mid h(\ell) = \ell \ \forall h \in H\}$

$$\mathrm{Fix}(M) \longleftarrow\!\!\! \dashv K \subseteq M \subseteq L$$
$$\text{subextension}$$

$$\mathrm{Fix}(M) = \{\sigma \in \mathrm{Gal}(L/K) \text{ s.t. } \sigma(\ell) = \ell \ \forall \ell \in M\}.$$

Give an inclusion-reversing injection between <u>closed</u> subgroups of $\mathrm{Gal}(L/K)$ and subextensions $K \subseteq M \subseteq L$.

(closed normal subgroups $\longleftrightarrow$ Galois subextensions).

<u>Exercise (*)</u>: Find a Galois extension L/K and a subgroup which is not closed.

$$\mathbb{C}((t)) \hookrightarrow \mathbb{C}((t^{1/\infty}))$$

$$\hat{\mathbb{Z}} \supseteq \mathbb{Z}$$

What is $\mathbb{C}((t^{1/\infty}))^{\mathbb{Z}}$

$$\mathbb{C}((t^{1/\infty}))^{\mathbb{Z}} = \mathbb{C}((t))$$

In general if H is any subgroup

$$L^H = L^{\overline{H}} \leftarrow \text{closure of } H$$

$$\mathrm{Fix}(L^H) = \overline{H}.$$

$\uparrow$ fixing ℓ is a closed condition on H

Same thing with $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}} \supseteq \mathbb{Z}$

[illegible] $\mathbb{Z}^{\times}$

$\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) = \mathbb{Z}_p$

Another nice example:

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots)/\mathbb{Q}_p$$

$$\prod_{p \text{ prime}} \mathbb{Z}/2\mathbb{Z} = \prod_{p \text{ prime}} \mathbb{F}_2$$

Exercise Find a non-closed subgroup.

$\mathbb{R}((t))$

Claim: If $L/\mathbb{R}((t))$ is an algebraic closure then L is the splitting field of $\{x^n - t\}_{n \geq 1}$

Fix this

$L \supseteq \mathbb{C}((t))$ (adjoining $\sqrt{-1}$)

$i \leftarrow i$

Fix in L a compatible system of roots of $x^n - t$

$t^{1/n}$ (for $n = mk$ $(t^{1/n})^m = t^{1/k}$).

$$\mathbb{C}((t^{1/\infty})) = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$$

Claim this is algebraically closed

$\mathrm{Gal}(\mathbb{C}((t^{1/\infty}))/\mathbb{R}((t)))$.

complex conjugation on coefficients

$\mathbb{Z}/2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \curvearrowright \mathbb{C}((t^{1/\infty}))$

$$\begin{tikzcd} & \cdot \arrow[dl, "\mathbb{Z}/2\mathbb{Z}"'] \arrow[dr, "\hat{\mathbb{Z}}(1) = \varprojlim \mu_N(\mathbb{C})"] & \\ \mathbb{R}((t^{1/\infty})) \arrow[dr] & & \mathbb{C}((t)) \arrow[dl, "\mathbb{Z}/2\mathbb{Z} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})"] \\ & \mathbb{R}((t)) & \end{tikzcd}$$

$$\hat{\mathbb{Z}}(1) \rtimes \cancel{\mathbb{Z}/2\mathbb{Z}}$$

$$\hat{\mathbb{Z}}(1) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$

Action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$
is just action of complex conjugation on roots of unity.

If L/K is a separable closure
then L/K is determined up to isomorphism
$\mathrm{Gal}(L/K) \leftarrow$ absolute Galois group of K G_K
is determined up to isomorphism.

not typically unique.

is unique iff the group is abelian.

Exercise:

Different choices of L/K $\leftrightsquigarrow$ conjugation of the Galois group

$$\begin{tikzcd} L_2 & L_1 \arrow[l, bend right, "\sim"', "F"] \\ K \arrow[u, "\theta"] \arrow[ur] & \end{tikzcd}$$

so

$$\mathrm{Gal}(L_1/K) \xrightarrow[\sim]{c_F} \mathrm{Gal}(L_2/K)$$

K

$\xrightarrow[e_g]{\sim}$

$e_g^{-1} \circ e_f =$ conjugation by $g^{-1} \circ f$

The absolute Galois group of $\mathbb{R}((t))$ is isomorphic to $\hat{\mathbb{Z}}(1) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$

<u>Theorem:</u> $G_{\mathbb{Q}_p}$ has a canonical quotient isomorphic to

$$\hat{\mathbb{Z}}^{(\ell)}(1) \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

$$\varprojlim_{(N,\ell)=1} \mu_N(\overline{\mathbb{F}}_p)$$

"Tame quotient" $\rightsquigarrow$ corresponding Galois extension is "maximal tamely ramified extension" of $\mathbb{Q}_p$.

The part of $G_{\mathbb{Q}_p}$ which fits the analogy $\mathbb{Q}_p \leftrightarrow$ Laurent series in the variable p

w/ coefficients in $\mathbb{F}_p$.

Fix $L/\mathbb{Q}_p$ a separable closure

then the maximal tamely ramified subextension is L^{tame} = splitting field of

$$\{x^n - p\}_{\substack{(n,p) \\ \text{coprime}}}$$

$\mathrm{Gal}(L^{tame}/L)$

$$\begin{tikzcd}
 & L^{tame} \arrow[dl, dash, "\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)"'] \arrow[dr, dash, "\mathbb{Z}^{\ell}(1)"] & \\
\bigcup\limits_{(n,p)\neq 1} \mathbb{Q}_p(p^{1/n}) & & \bigcup\limits_{(n,p)=1} \mathbb{Q}_p(\zeta_n) \\
 & \mathbb{Q}_p \arrow[ul, dash] \arrow[ur, dash, "\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)"'] &
\end{tikzcd}$$

Exercise ① Show L^{tame} contains ζ_p a primitive pth root of unity

② Find something not in L^{tame}

(Find a nontrivial Galois extension

of L^{fave}).