Last time: strict p-rings /

multiplicative representatives / formulas for addition + multiplication.

Punchline:

strict p-rings $\longleftrightarrow$ perfect rings in characteristic p

$$R \longmapsto \bar{R} = R/p$$

$$W(\bar{R}) \longleftarrow\!\shortmid \bar{R}$$

$$\parallel$$

$$\sum_{n=0}^{\infty} [a_n] p^n$$

with $[a]$ for a in $\bar{R}$

+ addition / mult in universal formulas.

2 ways to get $\mathbb{Q}_p$

$\mathbb{Q} \to \mathbb{Q}_p$ = completion for $|\ |_p$

$\mathbb{F}_p \to W(\mathbb{F}_p)[\frac{1}{p}]$

Generalizes e.g. by taking extensions of $\mathbb{Q}$, completing for extensions of $|\ |_p$

Start with any perfect ring $\bar{R}$

$W(\bar{R})[\frac{1}{p}]$.

Some overlap between these generalizations of $\mathbb{Q}_p$.
(but not a lot initially).
But they'll come back together.

Field (algebraic)
Extensions of $\mathbb{Q}_p$ (and other related fields).

$\overline{\mathbb{Q}_p}$ = "Laurent series in variable p" with coefficients in $\mathbb{F}_p$.
$\left(= W(\mathbb{F}_p)[\frac{1}{p}]\right)$.

Actual Laurent series rings.

$\mathbb{C}((t))$ = formal Laurent series with complex coefficients.
(formal functions on the punctured disk).

Example of a field extension:

$$\mathbb{C}((t^{1/n})) \hookleftarrow \mathbb{C}((t))$$

n is a positive integer.

Exercise $\mathbb{C}((t^{1/n})) \simeq \mathbb{C}((t))[x]/(x^n - t)$

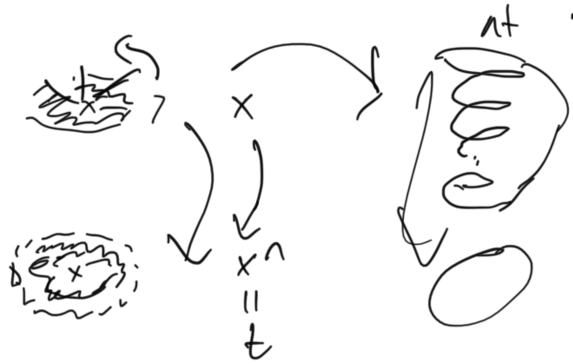
irreducible in $\mathbb{C}((t))$

$\uparrow$

Eisenstein's criterion at t.

coordinate x

coordinate t



x

$x^n = t$

Claim: This gives all finite extensions of $\mathbb{C}((t))$.

Exercise

Idea: Fundamental group of punctured disk is $\mathbb{Z}$.

so finite coverings $\leftrightarrow$ $\mathbb{Z} \to \mathbb{Z}/n$

$\updownarrow \sim$

$x \to x^n$.

$$\mathbb{C}((t^{1/\infty})) = \operatorname{colim} \mathbb{C}((t^{1/N}))$$

is an algebraic closure of $\mathbb{C}((t))$

$$\mathrm{Gal}\left(\mathbb{C}((t^{1/\infty}))/\mathbb{C}((t))\right) = \lim_{N} \mathrm{Gal}\left(\mathbb{C}((t^{1/N}))/\mathbb{C}((t))\right)$$

$$= \cancel{\lim_{N} \mathbb{Z}/N\mathbb{Z}}.$$

$$\hat{\mathbb{Z}}(1) \qquad = \lim_{N} \mu_N(\mathbb{C})$$

$$e^{2\pi i/n} \quad \mu_N(\mathbb{C}) = N \text{ th roots of unity in } \mathbb{C}.$$

$$\begin{array}{cc} \uparrow & \cong \\ 1 & \mathbb{Z}/N\mathbb{Z} \end{array}$$

$$\hat{=} \lim \mathbb{Z}/N\mathbb{Z} = \hat{\mathbb{Z}}$$

__Awesome__ — A complete description of the Galois group / all finite extensions.

An innocent difference between this & $\mathbb{Q}_p$:

$\mathbb{Q}_p$ = Laurent series in the variable p with coefficients in $\underline{\mathbb{F}_p}$.

$\mathbb{C}((t))$ = Laurent series in variable t with coefficients in $\mathbb{C}$.

$\mathbb{F}_p$ is not alg. closed but $\mathbb{C}$ is algebraically closed

<u>Example</u> $\underset{\in}{\mathbb{R}}((t)) \subseteq \underset{\in}{\mathbb{C}}((t))$ degree 2 extension

$$\left(\mathbb{R} \subseteq \mathbb{C} \quad \rightsquigarrow \quad \mathbb{R}((t)) \subseteq \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}((t)) \right)$$

<u>Exercise</u> Compute $\mathrm{Gal}\left(\mathbb{C}((t^{1/\infty})) / \mathbb{R}((t)) \right)$.

$$\mathbb{R}((t)) \overset{\mathbb{Z}/2}{\subseteq} \mathbb{C}((t)) \overset{\text{Galois}}{\subseteq} \mathbb{C}((t^{1/\infty}))$$

Galois

$$\mathbb{Z}/2 \ltimes \hat{\mathbb{Z}}(1)$$

$\hat{}$

$$\begin{tikzcd} \hat{\mathbb{Z}}(1) \arrow[r] & G \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[l, dashed, bend right] \\ & \mathbb{Z}/2\mathbb{Z} \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

$\langle \sigma \rangle$

σ = complex conjugation on coefficient.

action $\hat{\mathbb{Z}}(1)$

$$\sigma x \sigma^{-1} = -x$$

$x \in \varprojlim \mu_N(\mathbb{C})$

$= (\zeta_N)_N$ ← compatible choice of Nth roots of unity.

$$\sigma x \sigma^{-1} = (\overline{\zeta}_N)_N$$

$= (\zeta_N^{-1})_N$ ← mult. notation

$= -(\zeta_N)_N$ ← additive notn.

For $\mathbb{Q}_p$ we'll imitate the geometric picture: taking Nth roots of t ⟷ taking Nth roots of p with $(N, p) = 1$

p-adic Hodge theory.

= Wild inertia

$\rightsquigarrow$ kernel

$$G_{\mathbb{Q}_p} \to \mathbb{R} \to \mathbb{C} \qquad \longleftrightarrow \qquad \mathbb{F}_p \to \overline{\mathbb{F}_p}.$$

"Tame quotient"

$$1 \to \hat{\mathbb{Z}}^{(p)}(1) \to G \to \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \to 1$$

$$\hat{\mathbb{Z}}^{(p)}(1) = \lim_{(N,p)=1} \mu_N(\_\_)$$

Realizes going from $\mathbb{F}_p$ to $\overline{\mathbb{F}_p}$.

Going from $\mathbb{F}_p$ to $\overline{\mathbb{F}_p}$:

For power series:

$$\mathbb{F}_p((t)) \rightsquigarrow \overline{\mathbb{F}_p}((t)) = \overline{\mathbb{F}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_p((t$$

algebraic exten

? wrong

$p = 3 \qquad \mathbb{F}_9 \qquad \sqrt{-1}$

$$\sqrt{-1}(1 + t + t^2 + \cdots)$$

$$= \sqrt{-1} + \sqrt{-1}\,t + \sqrt{-1}\,t^2 + \cdots$$

If $f \in \overline{\mathbb{F}_p} \otimes \mathbb{F}_p((t))$

then $f = \sum_{i=1}^{M} a_i \otimes f_i$

$a_i \in \overline{\mathbb{F}_p} \qquad f_i \in \mathbb{F}_p((t))$

$$[\mathbb{F}_p(a_1, \dots, a_m) : \mathbb{F}_p] < \infty$$

$$f \in \mathbb{F}_p(a_1, \dots, a_m)((t))$$
$$\cap$$
$$\overline{\mathbb{F}_p}((t)).$$

$$\overline{\mathbb{F}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_p((t)) = \operatorname{colim}_n \mathbb{F}_{p^n} \otimes \mathbb{F}_p((t))$$
$$= \operatorname{colim}_n \mathbb{F}_{p^n}((t)) \subsetneq \overline{\mathbb{F}_p}((t))$$

$$\left(\mathbb{F}_{p^n} := \text{Roots of } x^{p^n} - x \text{ in } \overline{\mathbb{F}_p}. \right)$$

We'll see later: $\overline{\mathbb{F}_p}((t)) \supsetneq \overline{\mathbb{F}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_p((t))$

$\overline{\mathbb{F}_p}((t))$: very trancendental over $\mathbb{F}_p((t))$.

Dichotomy of extensions of $\mathbb{Q}_p$

$\mathbb{Q}_p$

$\mathbb{F}_p$

ext'n from coefficients

geometric part.

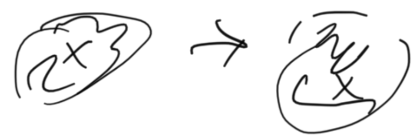
ext'n from geometric part

are unramified | are ramified

tame → prime to p

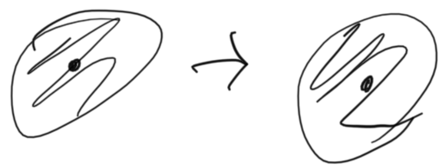
wild → p-power.

$t \to t^n$ as a map from



is a covering space.

$t \mapsto t^n$



ramified at 0 ~ pre-image of 0 has fewer than n points (in fact zero points)

differential

$n\, t^{n-1}\, dt$ vanishes at $t=0$.

Over $\mathbb{Q}_p$ ~ filling in disk

= adding back in $p=0$

$\mathbb{Z}_p$.

Ramification is about whats happening to $\mathbb{Z}_p$

esp. above $p=0$.