2020-09-10

Constructed $\mathbb{Q}_p$ ✓

Analytically ✓

Algebraically ✓

Geometrically ⇝ power series in variable p

$\widehat{\wr}$

Fix a set S of representatives for

$$\mathbb{Z}_p / p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$$

e.g. $S = \{0, 1, 2, \dots, p-1\}$

$S =$ multiplicative lifts $=$ Roots of $X^p - X$ in $\mathbb{Z}_p$. ($= 0$ & $(p-1)$st roots of unity)

$$\sum_{k \geq -N} a_k p^k \quad \text{for } a_k \in S.$$

Exercise: What is -1 for these two choices of representatives (pay attention $p=2$).

Question to address: How to resolve addition/mult back into these series?

Focus on multiplicative lifts ⇝ everything boils down to resolving addition.

Multiplicative lifts ~ elements with a p^nth root for any n.

- Generalize to strict p-rings

R is a strict p-ring if

1. R is p-adically complete.
2. p not nilpotent in R

   (equiv. R is p-torsion free / R is flat over $\mathbb{Z}$),

3. R/p is a perfect ring in characteristic p.

   $R/p =: \overline{R}$

   ($x \overset{F}{\mapsto} x^p$ is an isomorphism).

If R is a strict p-ring

then mult. section

$$\overline{R} \xrightarrow{[\cdot]} R \quad \text{is}$$

$$x \mapsto \lim_{n \to \infty} \left(\widetilde{x^{1/p^n}} \right)^{p^n} := [x]$$

$\widetilde{x^{1/p^n}}$ is any lift

of X^{1/p^∞}.

<u>Lemma</u>: If R is a strict p-ring then any $r \in R$ has a unique expansion

$$r = \sum_{i=0}^{\infty} [a_i] p^i$$

for $a_i \in \overline{R}$

<u>Question</u>: How to add/multiply these "power series"

(How do we resolve e.g. $[a_1] + [a_2]$ into a power series)

<u>Example</u>: ① $\mathbb{Z}_p$ is a strict p-ring.

② $\mathbb{Z}_3[x]/x^2+1$ is a strict p-ring with residue field $\mathbb{F}_9$.

(3-adically complete, $(\mathbb{Z}_?[x]/x^2+1)/(?)$ $= \mathbb{F}_3[x]/x^2+1$ field w/ 9 elements)

polynomials with coeffs in $\mathbb{Z}$ and exponents $\mathbb{Z}[\frac{1}{p}]$

③ $\left(\mathbb{Z}[X^{1/p^\infty}]\right)^\wedge$ p-adically complete $R \to \lim R/p^n$.

Residue field is $\boxed{\mathbb{F}_p[X^{1/p^\infty}]}$

perfect ring (perfection of $\mathbb{F}_p[x]$).

<u>Exercise</u>: What is the universal property of $\mathbb{Z}[x^{1/p^\infty}]$

$\mathbb{Z}[x^{1/p^\infty}]^\wedge$?

$$\mathrm{Hom}_{\text{Rings}}(\mathbb{Z}[x^{1/p^\infty}], R)$$

$$= \{(a_0, a_1, a_2, \ldots) \in R^{\mathbb{Z}_{\geq 0}} \text{ s.t. } a_i^p = a_{i-1} \text{ for } i \geq 1\}$$

$$= \lim R \xleftarrow{F^p} R \xleftarrow{F^p} R \leftarrow \ldots$$

$$\mathrm{Hom}_{\substack{p\text{-adically} \\ \text{complete rings}}}(\mathbb{Z}[x^{1/p^\infty}]^\wedge, R)$$

= same things

<u>Example</u> a, b in $\overline{R}$

$[a] + [b]$ mod p

$= a + b$ so

$$\underline{[a] + [b]} = [a+b] + O(p)$$

To see what's in $O(p)$

let's compute $[a+b]$ using representatives adapted to $[a] + [b]$

comparing with

$[a^{1/p^n}] + [b^{1/p^n}]$ is a lift of $((a+b)^{1/p^n})$.

$$\text{so } [a+b] = \lim_{n\to\infty}\left([a^{1/p^n}] + [b^{1/p^n}]\right)^{p^n}$$

$$= [a] + [b] + \left(\sum_{k=1}^{p-1} \binom{p}{k}\Big/p \; [a^{k/p} b^{p-k/p}]\right) p \; + \left(\sum_{\substack{k=0\\(p,k)=1}}^{p^2-1} \binom{p^2}{k}\Big/p^2 \; [a^{k/p^2} b^{p^2-k/p^2}]\right) p^2 + O(p^3).$$

Integers

So:

$$[a] + [b] = [a+b] + \left[-\sum_{k=1}^{p-1} \binom{p}{k}\Big/p \; a^{k/p} b^{p-k/p}\right] p + O(p^2)$$

$p \neq 2$.

Complicated to keep going: need to keep track (but possible) of correction from replacing formulas in R with formulas in $\bar{k}$

Important qualitative aspects:

To compute the coefficient $c_n \in \bar{R}$ of p^n in $[a] + [b] = \sum [c_n] p^n$

~~only depends on~~

is a polynomial in a^{1/p^n}, b^{1/p^n}

Theorem: There are universal polynomials

$$s_n, m_n \in \mathbb{F}_p[x_0, x_1, x_2, \dots, x_n, y_0, \dots, y_n]$$

s.t. for any strict p-ring R

and for

$$a = \sum_{n=0}^{\infty} [a_n] p^n \qquad b = \sum [b_n] p^n$$

we have

$$a + b = \sum_{n=0}^{\infty} \left[s_n(a_0^{1/p^n}, a_1^{1/p^{n-1}}, \dots, a_n, b_0^{1/p^n}, b_1^{1/p^{n-1}}, \dots, b_n) \right] p^n$$

and similar

$$ab = \sum_{n=0}^{\infty} [M_n(\ldots)] p^n.$$

Corollary The functor

$$R \to \bar{R} = R/p$$

is an equivalence of categories between perfect $\mathbb{F}_p$-algebras and strict p-rings.

Simple / Exercise: $R \to \bar{R}$ is faithful.

Fully faithful / essentially surjective can verify using these formulas:

Construct a functor $\bar{R} \to W(\bar{R})$

perfect $\mathbb{F}_p$-algebras $\to$ strict p-rings.

$\bar{R} \longrightarrow$ set of formal series $\sum_{n=0}^{\infty} [a_n] p^n$ for a_n in $\bar{R}$

with addition + multiplication defined using polynomials S_n and M_n

(need to check you get a commutative ring).

As a set $W(\bar{R}) = \prod_{n=0}^{\infty} \bar{R}$

$n=0$

~~we don't use~~ $\sum [a_n] p^n \longleftrightarrow (a_n)$

Instead use $\sum [a_n] p^n \longleftrightarrow (a_n^{p^n})$

(If you use this normalization the addition multiplication don't involve roots),

Because

$$(a_n) + (b_n) = (S_n(a_0, a_1, \ldots, a_n, b_0, b_1, \ldots b_n))$$

This defines a ring $W(\overline{R})$ when $\overline{R}$ is any $\mathbb{F}_p$-algebra.

Moreover can actually define $W(R)$ for any ring R

(W functor of p-typical Witt vectors).

Hazewinkel's book best reference
Rabinoff also good (good for NT).
Serre's Local fields

$$\mathbb{Z}[x_0^{1/p^\infty}, x_2^{1/p^\infty}, x_3^{1/p^\infty}, \ldots, y_0^{1/p^\infty}, y_3^{1/p^\infty}, \ldots]^\wedge \rightsquigarrow R_{univ}$$

$\to \mathbb{F}_p[\bar{x}_0^{1/p^\infty}, \ldots$

$\mathbb{Z}$ strict p ring. $\bar{y}^{1/p^\infty}, \ldots]$

Addition + multiplication laws are universal here.

Suppose R is a strict p-ring.

$a = \sum_{n=0}^{\infty} [a_n] p^n$ $\quad b = \sum_{n=0}^{\infty} [b_n] p^n$

$a_n, b_n \in \bar{R}$.

Unique map from

$$R_{univ} \to R$$
$$x_i \mapsto [a_i]$$
$$\overset{\shortparallel}{[\bar{x}_i]} \quad (x_i^{1/p} \to [a_i^{1/p}] = [a_i]^{1/p}),$$
$$y_i \mapsto [y_i].$$

sends

$$\sum [\bar{x}_i] p^n = \sum x_i p^n \to \sum [a_i] p^n$$
$$\sum [\bar{y}_i] p^n = \sum y_i p^n \to \sum [b_i] p^n$$

Suffices to give formulas for adding & multiplying these.

Last ingredient: p-divisibility properties of multinomial coefficients.

+ induction

Include as exercise

w/ more details in the notes.