Last time: Computation that $H^1(\mathbb{C}^*, \mathbb{Z})$
"same" as $H^2(\mathbb{P}^1, \mathbb{Z})$.
will put in notes.

Recall k is a field X/k alg. variety.

Define cohomology theories

- étale cohomology
- algebraic de Rham cohomology

Fix a prime ℓ
ℓ-adic étale cohomology
$\mathbb{Q}_\ell$ - vector space ~ geometric
w/ action of $\mathrm{Gal}(\bar{k}/k)$ ~
arithmetic.

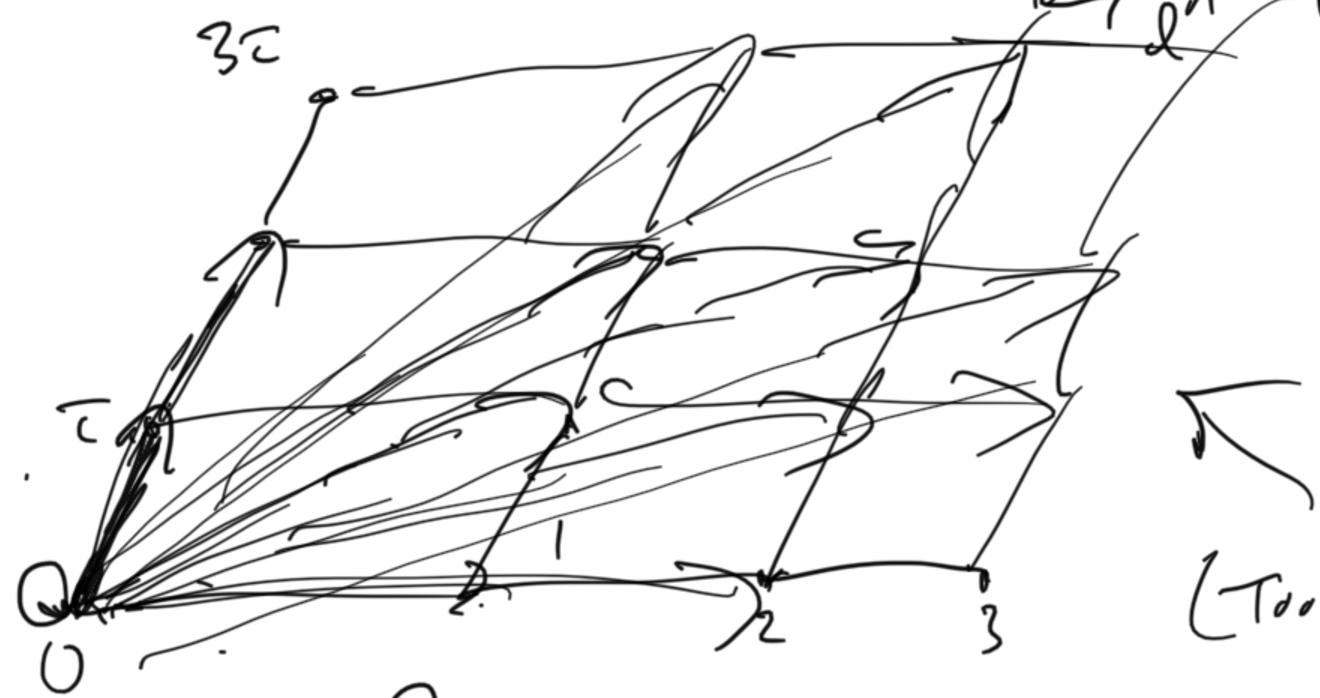
Example: $k = \mathbb{C}$ $E_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

$$H_1(E_\tau, \mathbb{Z}) = \mathbb{Z}\cdot\gamma_1 + \mathbb{Z}\cdot\gamma_\tau$$

$(\tau \in \mathbb{C} \setminus \mathbb{R})$

$$H_1(E_\tau, \mathbb{Z}/\ell^n\mathbb{Z}) = H_1(E_\tau, \mathbb{Z})/\ell^n H_1(E_\tau, \mathbb{Z})$$

$$\cong \mathbb{Z}/\ell^n \cdot \gamma_1 + \mathbb{Z}/\ell^n \cdot \gamma_\tau.$$

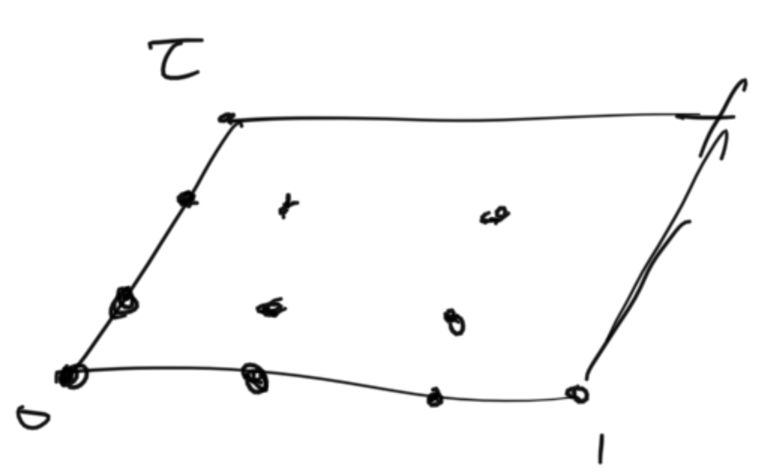


(Too many arrows)

Line segments give all of the classes in

$$H_1(E_\tau, \mathbb{Z}/3\mathbb{Z}).$$

If I don't scale up



9 dots representing // homology classes,

points P on E_τ

s.t. $3P = 0$.

$\wr$

Kernel of mult. by 3

on the group E_τ.

$\wr$

$E_\tau[3]$

In general can identify

$$H_1(E_\tau, \mathbb{Z}/\ell^n\mathbb{Z})$$

$$\cong E_\tau[\ell^n]$$

with $E_\tau[n]$

Weierstrass: meromorphic function $\wp$
periodic on the lattice $\mathbb{Z}+\mathbb{Z}\tau$

satisfying a differential equation

$$(\wp')^2 = \wp^3 + a\wp + b$$

$$E_\tau \longrightarrow y^2 = x^3 + ax + b \sqcup \infty$$

$$z \longmapsto (\wp'(z), \wp(z))$$

This type of equation makes sense over any field.

Group structure is given polynomials

In particular

$\cdot \ell : E_\tau \to E_\tau$ is given by
$z \to \ell z$

$$(x, y) \downarrow (p_\ell(x,y),\ q_\ell(x,y))$$

We can define $\underset{\text{Elliptic curve over } K}{\uparrow}$

$$\text{"}H_{1,\acute{e}t}(E, \mathbb{Z}/\ell^n)\text{"} := E[\ell^n](\overline{K}).$$

$$\circlearrowleft$$
$$Gal(\overline{K}/K)$$

by action on coordinates, preserves ℓ^n-torsion because p_{ℓ^n} q_{ℓ^n}

have coefficients in K

1

$$H^1_{et}(E, \mathbb{Q}_\ell) = \left(\left(\varprojlim E[\ell^n](\bar{k}) \right) [\tfrac{1}{\ell}] \right)^*$$

$$\cong \mathbb{Q}_\ell^2 \curvearrowleft \mathrm{Gal}(\bar{k}/k)$$

Algebraic varieties / $\mathbb{Q}$ $\rightsquigarrow$ $\mathbb{Q}_\ell$-representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

$X \rightsquigarrow H^i_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)$

Tate conjecture: Fully faithful on motives / $\mathbb{Q}$
(with $\mathbb{Q}_\ell$-coefficients).

Question: Which Galois representations come from motives?

$G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ $\qquad$ $G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

For every prime p

$$G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$$

(depends on choice up to conjugation)

Any (continuous) representation of $G_{\mathbb{Q}}$ is determined by its restrictions to $G_{\mathbb{Q}_p}$ for all p.

$$\underset{\substack{\text{\rotatebox{90}{$\cong$}}\\ \text{Inertia}}}{I} \subseteq G_{\mathbb{Q}_p} \twoheadrightarrow G_{\mathbb{F}_p} \leftarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p),$$

Simple.

Has a simple part and a complicated part.

$\mathbb{Q}_p$ "=" Laurent series in p

"=" functions on a punctured disk.
(formal)

$$\mathbb{C}((t))$$

$\cancel{\hat{\mathbb{Z}}} \leftarrow$ circle topologically

Galois group (basically) Fundamental group of a circle $\mathbb{Z}$.

Representations are simple!

This analogy is basically right when look at representations of $G_{\mathbb{Q}_p}$ on $\mathbb{Q}_\ell$-vector spaces when $\ell \neq p$.

Fontaine-Mazur conjecture:

The $\mathbb{Q}_\ell$-rep's coming from algebraic geometry instead are the ones such that:

① For almost all $p \neq \ell$, the restriction to

$G_{\mathbb{Q}_p}$ is unramified (factors through $G_{\mathbb{F}_p}$).

② A condition when $p = \ell$ that is more complicated.

describe & precisely using p-adic Hodge theory.

Family of smooth proper varieties parameterized by the punctured disk.

Looking at how the fundamental group $\hat{\mathbb{Z}}$ acts on cohomology of the fibers.

If you can fill in the puncture with a smooth proper variety then the action is trivial.

Monodromy over whole disk, contractible.

$\hat{\mathbb{Q}}$

For $G_{\mathbb{Q}_p}$ acting on ℓ-adic étale cohomology, filling in the puncture = giving equations w/ $\mathbb{Z}_p$-coefficients insted of $\mathbb{Q}_p$-coeffs

Filling in w/something smooth means you get a smooth projective variety over $\mathbb{F}_p$.

This breaks down when $\ell = p$; even if you can fill in the puncture nicely, the representation is not unramified.

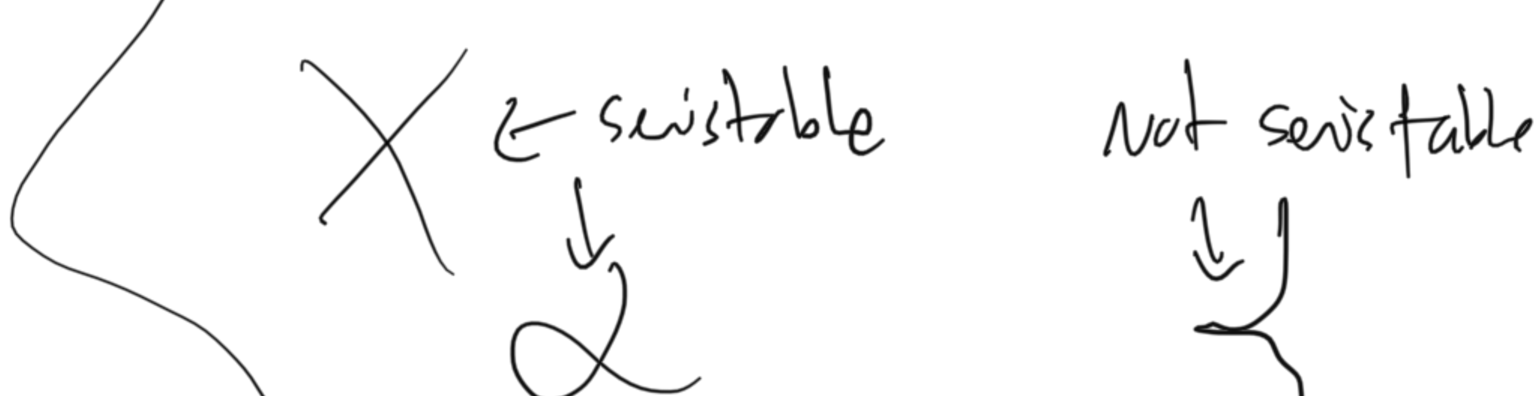
↳ Verify later explicitly w/ examples from elliptic curves.

One of the origins of p-adic Hodge theory: understand what nice properties this representation has even though it is ramified.

More generally ~ can't fill in the disk w/ something smooth

Expect to be able to arrange semistable reduction

↳ fill in fibers w/ something w/ nice singularities



⟼ having potentially unipotent monodromy

↑ when $\ell \neq p$ this forced by the structure of Galois group.

But need some p-adic HT
condition when $\ell = p$.

$\rightsquigarrow$ potentially semistable
representation of $G_{\mathbb{Q}_p}$

this is the condition
in the Fontaine-Mazur
conjecture

Why "p-adic Hodge theory"

$\rightsquigarrow$ have semistable reduction ~ equip
the de Rham cohomology with
a Frobenius and N
reflects unipotent monodromy.

$$G_{\mathbb{Q}_p} \circlearrowright H^i_{et}(X, \mathbb{Q}_p) \longleftrightarrow H^i_{dR}(X)$$ w/ Hodge filtration
+ Frobenius + N.

(In Hodge theory get an integration $\cong$
between singular & deRham cohomology after $\otimes \mathbb{C}$),

Here you tensor with Fontaine's period ring
B_{pst} $\rightsquigarrow$ enormous
has action of $G_{\mathbb{Q}_p}$,
Frob, N, and a
filtration.

p-adic HT à la Fontaine. ← arithmetic theory

(also p-adic HT à la Tate)

$\Updownarrow$ ≃ closer to actual Hodge theory

geometric theory / over $\mathbb{C}_p$

Really only properly understood in the last 10 years. ← work of Scholze, Fargues–Fontaine, Kedlaya.

gives the right language to understand this.