Hi! Math 6370 (2020-08-25)

Today & probably next time: Motivation

Hodge theory: The study of extra structure on the cohomology of complex algebraic varieties (or complex Kahler manifolds).

Example

\( \mathbb{C}^* \) - Punctured complex plane

Homotopic to a circle around the origin.

\[ H_1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z} \left[ \frac{\partial}{\partial t} \right] \]

\[ \phi(t) = e^{2\pi i t} \]

\[ \phi: \mathbb{C}, \{0, 1\} \to \mathbb{C}^* \]

Differential form \( \frac{dz}{z} \) is holomorphic.

\[ \oint_{\phi} \frac{dz}{z} = 2\pi i. \]

Local anti-derivative is \( \log z \), space coordinate \( \downarrow \downarrow \)

\( \frac{dz}{z} \) represent a element of \( H^1(\mathbb{C}^*, \mathbb{C}) \)

\[ \text{Hom}(H_1(\mathbb{C}^*, \mathbb{Z}), \mathbb{C}) \]

\[ \frac{dz}{z} \rightarrow 2\pi i \in \mathbb{C} \]
2. Fix $t \in \mathbb{C} \setminus \mathbb{R}$ (e.g. $t = e^{2\pi i/3}$).

$E_t = \mathbb{C} / (\mathbb{Z} \circ Z_t)$

Topologically (diffeomorphic)

$S^1 \times S^1 = \text{torus}$

$H_1(E_t, \mathbb{Z}) = \mathbb{Z}[\gamma_1] + \mathbb{Z}[\gamma_2] = \mathbb{Z}^2$

$\gamma_1$ parameterizes the line segment from 0 to 1

$\gamma_2$ parameterizes the line segment from 0 to 2

$dz \quad \frac{dz}{dx + idy} = \bar{z} = x + iy$.

Integrating over elements in

$H^1(E_t, \mathbb{C}) = \text{Hom}(H_1(E_t, \mathbb{Z}), \mathbb{C})$

$Sdz \rightarrow (1, \mathbf{t})$

$Sd\bar{z} \rightarrow (1, \mathbf{t})$.

Hodge theorem: Remembering the relation between singular and de Rham cohomology.
actually give more information than just remembering one or the other + the fact that they're from opques, we have an explicit isomorphism, integration given by integration.

Singular cohomology: $H^i_{\text{sing}}(X, \mathbb{C}) \sim H^i_{\text{deR}}(X, \mathbb{C})$

Special about $H^i_{\text{sing}}(X, \mathbb{C}) \cong H^i_{\text{sing}}(X, \mathbb{R}) \otimes \mathbb{C}$

inside some lattice.

Special about de Rham cohomology where a natural filtration, Hodge filtration a from holomorphic differential forms.

$\text{Fil}^p H^i_{\text{deR}}(X, \mathbb{C})$

- classes represented by differential forms with at least $p$ holomorphic variables.

Consider if $z_1, \ldots, z_n$ are local holomorphic coordinates then $z_1, \overline{z}_1, z_2, \overline{z}_2, \ldots$ are local real coordinates.

$\text{Fil}^1$ things like $f(z) \ dz \wedge d\overline{z} \wedge d\overline{z}_2 \wedge d\overline{z}_3$

Hodge structure is a $\mathbb{Z}$-module $M$ + a filtration $\mathbb{Z}$-singular cohomology) on $M \otimes \mathbb{C}$

( + some axiom relating the filtration...
In Example 2 above
our \( \mathbb{Z} \)-module was \( \mathbb{Z}^2 \)

\[
H^1(\mathbb{E}_c, \mathbb{Z})
\]
(dual basis for the paths \( \delta_1, \delta_2 \))

Filtration on \( \mathbb{Z}^2 \otimes \mathbb{C} \)

\[
\langle \delta_2 \rangle = \langle (1, 1) \rangle
\]

Remark For most choices of \( \tau \in \mathbb{C} \)

\( E_{\tau} \neq E_{\tau'} \) as a complex manifold,

but \( E_{\tau} \cong S_1 \times S_1 \cong E_{\tau'}, \) diffeomorphic,

This is exactly detected by the Hodge structure.

So Hodge structures really
detect (integral) intersection.

**Example:** Hodge conjecture says can detect

homology classes represented by algebraic varieties

using Hodge structures.

**Motives** Idea: If you "linearize" algebraic varieties

over a field \( K \), then they break down

into elemental chunks called motives.
which encode fundamental geometric & arithmetic information
over an algebraically closed field over a non-algebraically closed field

Little bit like taking & a finite group, 
& a finite \( G \)-set, then linearize 
by taking Maps \((X, G)\)

\( G \)
decompose into irreducible representations for \( G \).
E.g. \( \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \) is irreducible as a \( \mathbb{Z} \)-set.

but Maps \((X, G)\)
breaks up into \( n \) 1-dim \( L \) subrepresentations

\[ \text{character } 1 \rightarrow e^{2\pi i k/n} \quad k = 0, \ldots, n-1. \]

These clearly can be "seen" in cohomology, and move around via duals of \( \mathbb{C}^* \). E.g.
\[ \mathbb{C}^* = \mathbb{R}^+ \setminus \{0, \infty\} \]

\( H^2(\mathbb{P}^1, \mathbb{R}) \) and \( H^1(\mathbb{C}^*, \mathbb{R}) \)

I'll come back next time.

Moreover correctly supported cohom + non-compactability

Matrices are built out of 
smooth projective varieties
Example 1.1.

You can realize logarithm as extension of rational numbers
\[ \mathbb{R}^\times \cong \text{trivial} \quad \mathbb{Z}(-1) \cong H^2(\mathbb{P}^1) \]

1) Lack of definition of exact Hodge structure, make them precise
2) Realize it geometrically by gluing 2 points together in \( \mathbb{C}^\times \) (can shear this in nodal curve w/ 2 punctures).

Restate the Hodge conjecture as:

the map from motives (with \( \mathbb{Q} \)-coefficients) to Hodge structure is fully faithful.

Note: It's an difficult problem of

fundamental importance to understand

exactly which Hodge structures

are motivic = cone from algebraic variation.

While mathematical worlds turn out of special cases in

e.g., theory of Shimura varieties - the one type

of motivic Hodge structure that can possibly vary

in a suitable moduli space.
Varieties over $\mathbb{Q}$.

There are purely algebraically defined cohomology theories:

1. Algebraic de Rham cohomology
   - Use algebraic differential forms

2. $l$-adic étale cohomology ($l$ is a prime number)
   - $l$-adic numbers
   - $\mathbb{Q}_l$-vector space
   - $l$-adic numbers
   - $\left(\lim \mathbb{Z}_l(x) \left[ \frac{1}{x} \right]\right)$
   - "Laurent series in the variable $l^{-1}$"
   - $l^{-1} + 1 + 2l + 7l^2 + \ldots$

\[ \uparrow \]

Representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Over $\mathbb{Q}$, Algebraic de Rham $\cong$ de Rham

$l$-adic étale $\cong$ singular w/ $\mathbb{Q}_l$-coefficients

Conjecture (Tate): $\text{Mat}(n)/\mathbb{Q} \rightarrow l$-adic étale cohom. $\rightarrow$

(l/w $\mathbb{Q}_l$-coeff) $l$-adic reps of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

is fully faithful.

Question: Which Galois representations come from motives?

$p$-adic Hodge theory starts to show up

Fontaine-Mazur conjecture
Specific instars include an adult of Ferrate.