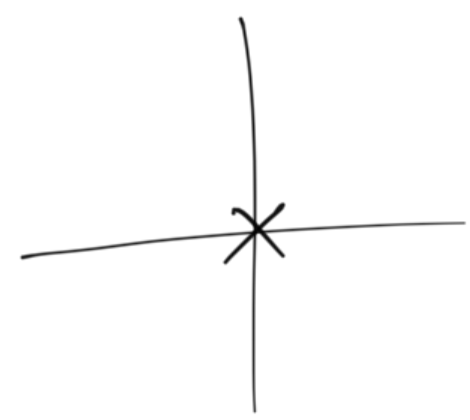
Hi! Math 6370 (2020-08-25)

Today + Probably next time: Motivation

Hodge theory: The study of extra structure on the cohomology of complex algebraic varieties (or complex Kahler manifolds).

Example

(1) $\mathbb{C}^*$ ← Punctured complex plane



Homotopic to a circle around the origin.

$$H_1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z}[\gamma]$$

$$\gamma(t) = e^{2\pi i t}$$

$$\gamma : [0,1] \to \mathbb{C}^*$$

Differential form $\frac{dz}{z}$ ← holomorphic.

$$\int_\gamma \frac{dz}{z} = 2\pi i.$$

local anti-derivative is $\log z$.

$$\frac{dz}{z} \text{ represent an element of } H^1(\underbrace{\mathbb{C}^*}_{\text{space}}, \underbrace{\mathbb{C}}_{\text{coefficients}})$$

$$= \mathrm{Hom}(H_1(\mathbb{C}^*, \mathbb{Z}), \mathbb{C})$$

$$\cong \mathbb{C}$$

$$\frac{dz}{z} \rightsquigarrow 2\pi i \in \mathbb{C}$$

② Fix $\tau \in \mathbb{C}\setminus\mathbb{R}$ (e.g. $\tau = e^{2\pi i/3}$).

$$E_\tau = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$$

Topologically (diffeomorphism) $\rightsquigarrow$ $S^1 \times S^1$ = torus

$$H_1(E_\tau, \mathbb{Z}) = \mathbb{Z}[\gamma_1] + \mathbb{Z}[\gamma_\tau] = \mathbb{Z}^2$$

γ_1 parameterizes the line segment from 0 to 1

γ_τ parameterizes the line segment from 0 to τ

$$\underset{dx+idy}{\overset{\shortparallel}{dz}} \qquad \underset{dx-idy}{\overset{\shortparallel}{d\bar{z}}} \qquad (\bar{z} \text{ is complex conjugate}). \quad z = x+iy.$$

Integrating gives elements in

$$H^1(E_\tau, \mathbb{C}) = \mathrm{Hom}(H_1(E_\tau, \mathbb{Z}), \mathbb{C})$$

$$\int dz \to (1, \tau)$$

$$\int d\bar{z} \to (1, \bar{\tau}).$$

Hodge theory: Remembering the relation between singular & de Rham cohomology

singular ← paths/chains

de Rham ← differential forms

actually give more information than
just remembering one or the other + the fact
that they're isomorphic.

we have an explicit isomorphism
given by integration.

Singular cohomology: $H^i_{sing}(X, \mathbb{C}) \xrightarrow[\sim]{\text{integration}} H^i_{dR}(X, \mathbb{C})$

Special about $H^i_{sing}(X, \mathbb{C}) \cong H^i_{sing}(X, \mathbb{Z}) \otimes \mathbb{C}$

some lattice inside.

Special about de Rham cohomology ~ have a natural filtration
Hodge filtration ~ from holomorphic differential forms.

$$\mathrm{Fil}^p H^i_{dR}(X, \mathbb{C})$$

classes represented by differential forms with at least p holomorphic variables.

(i.e. if $z_1, \ldots, z_m$ are local holomorphic coordinates then
$z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots$ are local
real manifold coordinates,
Fil^2 things like

$$f(\ldots)\, dz_1 \wedge dz_2 \wedge d\bar{z}_3$$

Hodge structure is a $\mathbb{Z}$-module M + a filtration
(singular cohomology) on $M \otimes \mathbb{C}$

(+ some axioms relating the filtration

to its conjugate)

In Example 2 above

our $\mathbb{Z}$ module was $\mathbb{Z}^2$
$$\mathbb{Z}^2 = H^1(E_\tau, \mathbb{Z})$$
(dual basis for the paths γ_1, γ_τ)

Filtration on $\mathbb{Z}^2 \otimes \mathbb{C}$
is $\langle dz \rangle = \langle (1, \tau) \rangle$

<u>Remark</u> For most choices of τ & τ'
$E_\tau \not\cong E_{\tau'}$ as a complex manifold.
but $E_\tau \cong S_1 \times S_1 \cong E_{\tau'}$ diffeomorphic.

This is exactly detected by the Hodge structure.

So Hodge structures really detect interesting information.

<u>Example</u>: Hodge conjecture says can detect homology classes represented by algebraic varieties using Hodge structures.

<u>Motives</u> Idea: If you "linearize" algebraic varieties over a field K, then they break down into elemental chunks called motives.

which encode fundamental geometric & arithmetic information,

over an algebraically closed field

over a non-algebraically closed field,

Little bit like taking G a finite group, X a finite G-set, then linearize by taking $\mathrm{Maps}(X, \mathbb{C})$

$G \circlearrowright$

decompose into irreducible representations for G.

E.g. $\mathbb{Z} \circlearrowright \mathbb{Z}/n\mathbb{Z}$ ⤳ irreducible as a $\mathbb{Z}$-set.

but $\mathrm{Maps}(X, \mathbb{C})$ breaks up into n 1-dim'l subrepresentations

character $1 \mapsto e^{2\pi i k/n}$ $k = 0, \ldots, n-1$.

These chunks can be "seen" in cohomology, and move around using standard operations. E.g.

$$\underbrace{\mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}}$$

$$\underbrace{H^2(\mathbb{P}^1, \mathbb{Z}) \text{ and } H^1(\mathbb{C}^*, \mathbb{Z})}_{\text{same motive}}$$

I'll come back next time. ⤳ ⟵ Use compactly supported cohom + Poincaré duality

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~~Motives~~ Motives are built out of smooth projective varieties

$\hookleftarrow$ so often restrict to these,

Example (Exercise) 4.3.1

. You can realize logarithms as extensions of motives

$\mathbb{Z} \rightsquigarrow$ trivial $H^0(pt)$ $\qquad \mathbb{Z}(-1) \rightsquigarrow H^2(\mathbb{P}^1)$.

1) Look up def'n of mixed Hodge structures, make this precise

2) Realize it geometrically by gluing 2 points together in $\mathbb{C}^*$ (can show this a nodal cubic w/ 2 punctures).

Restate the Hodge conjecture as:

The map from motives/$\mathbb{C}$ (with $\mathbb{Q}$-coefficients) to Hodge structures is fully faithful.

Note: It's an ~~difficult~~ intractable problem of fundamental importance to understand exactly which Hodge structures are motivic = come from algebraic varieties.

Σ

Whole mathematical worlds born out of special cases $\rightsquigarrow$ e.g. theory of Shimura Varieties $\rightsquigarrow$ the one type of motivic Hodge structure that can possibly vary in a simple moduli space.

Varieties over $\mathbb{Q}$.

There are purely algebraically defined cohomology theories.

1. Algebraic de Rham cohomology
   - Use algebraic differential forms
2. ℓ-adic étale cohomology (ℓ is a prime number)

   gives $\mathbb{Q}_\ell$-vector space

   ℓ-adic numbers

   $$\left((\lim \mathbb{Z}/\ell^n) \left[\tfrac{1}{\ell} \right] \right).$$

   "Laurent series in the variable ℓ"

   $$\ell^{-1} + 1 + 2\ell + 7\ell^2 + \dots$$

Representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Over $\mathbb{C}$, Algebraic de Rham $\cong$ analytic de Rham

ℓ-adic étale $\cong$ singular w/ $\mathbb{Q}_\ell$-coefficients

Conjecture (Tate): $Mot/\mathbb{Q}$ (w/ $\mathbb{Q}_\ell$-coeffs) $\to$ ℓ-adic étale cohomology. ℓ-adic rep's of $Gl(\overline{\mathbb{Q}}/\mathbb{Q})$

is fully faithful.

Question: Which Galois representations come from motives?

p-adic Hodge theory starts to show up.

Fontaine-Mazur conjecture

Specific instances include Wiles's proof of Fermat.