Hello. Is this legible? Week 2, Th (1/20)

Continued fractions + solving the gcd equation.

a, b $ax + by = \gcd(a,b)$.

Quiz something like

Express $[1;2,3,4]$ as a reduced fraction.

$$[1;2,3,4] = 1 + \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}$$ Continued fraction.

$$= 1 + \cfrac{1}{2 + \cfrac{1}{\frac{13}{4}}} = 1 + \cfrac{1}{2 + \frac{4}{13}}$$

$$= \frac{a}{b}$$

Exercise 2 on worksheet

Recall: A real number α is rational $\iff$ the continued fraction expansion of α is finite. $\alpha = [a_0; a_1, \ldots, a_m]$.

Can use this o show a number is rational or irrational

Example: $\sqrt{2} = [1; 2, 2, 2, 2, \ldots]$ is infinite so $\sqrt{2}$ is irrational.

Expanded Example: $\left(= 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}} \right)$

How to show $\sqrt{2} = [1; 2, 2, 2, \dots]$

Follow algorithm! (See Th. lecture or accompanying notes)

① $\sqrt{2} = \lfloor \sqrt{2} \rfloor + (\sqrt{2} - \lfloor \sqrt{2} \rfloor)$

$= 1 + (\sqrt{2} - 1)$

$\sqrt{2} = 1.4\dots$ so $\lfloor \sqrt{2} \rfloor = 1$

② $\sqrt{2} - 1 = \dfrac{1}{1/(\sqrt{2}-1)}$ $\quad 1/(\sqrt{2}-1) = \dfrac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \dfrac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$

$\left\lfloor \dfrac{1}{\sqrt{2}-1} \right\rfloor = \lfloor \sqrt{2}+1 \rfloor = \underline{2}$. $\quad (\sqrt{2}+1 = 2.4\dots)$.

so $\sqrt{2} = 1 + \dfrac{1}{2 + (\sqrt{2}+1-2)} = 1 + \dfrac{1}{2 + (\sqrt{2}-1)}$

③ $\sqrt{2} - 1 = \dfrac{1}{1/(\sqrt{2}-1)}$ $\quad \dfrac{1}{(\sqrt{2}-1)} = \dots$ get same thing! So computation repeats.

Using the table

$\alpha = \sqrt{2}$ $\quad\quad$ $\alpha = [a_0; a_1, a_2 \ldots]$ $\quad\quad$ This entry determines the next row completely

$\alpha_0 = \sqrt{2}$

i	$a_i = \lfloor \alpha_i \rfloor$	$\beta_i = \alpha_i - \lfloor \alpha_i \rfloor$	$\alpha_{i+1} = \frac{1}{\beta_i}$
0	1	$\sqrt{2} - 1$	$\frac{1}{\sqrt{2}-1} = \sqrt{2} + 1$
1	2	$\sqrt{2} - 1$	$\frac{1}{\sqrt{2}-1} = \sqrt{2} + 1$
2			
3			

Exercise 1 - (3)

Find all integer solutions of $ax + by = \gcd(a,b)$ (first three parts).

In video – can find <u>one</u> solution by unraveling Euclidean algorithm.

<u>Example</u> $a = 60$ $b = 22$ $\gcd(a,b) = 2$

$$60 = 2 \cdot 22 + 16$$
$$22 = 1 \cdot 16 + 6$$
$$16 = 2 \cdot 6 + 4$$
$$6 = 1 \cdot 4 + 2$$
$$4 = 2 \cdot 2 + 0$$

$$2 = 1 \cdot 6 - 1 \cdot 4$$
$$2 = 1 \cdot 6 - 1(1 \cdot 16 - 2 \cdot 6)$$
$$2 = 3 \cdot 6 - 1 \cdot 16$$
$$2 = 3 \cdot (1 \cdot 22 - 1 \cdot 16) - 1 \cdot 16$$
$$2 = 3 \cdot 22 + (-4) \cdot 16$$
$$2 = 3 \cdot 22 + (-4) \cdot (60 - 2 \cdot 22)$$
$$2 = 11 \cdot 22 + (-4) \cdot 60$$

$$a \; x + b \; y = \gcd(a,b).$$
$$60(-4) + 22(11) = 2$$

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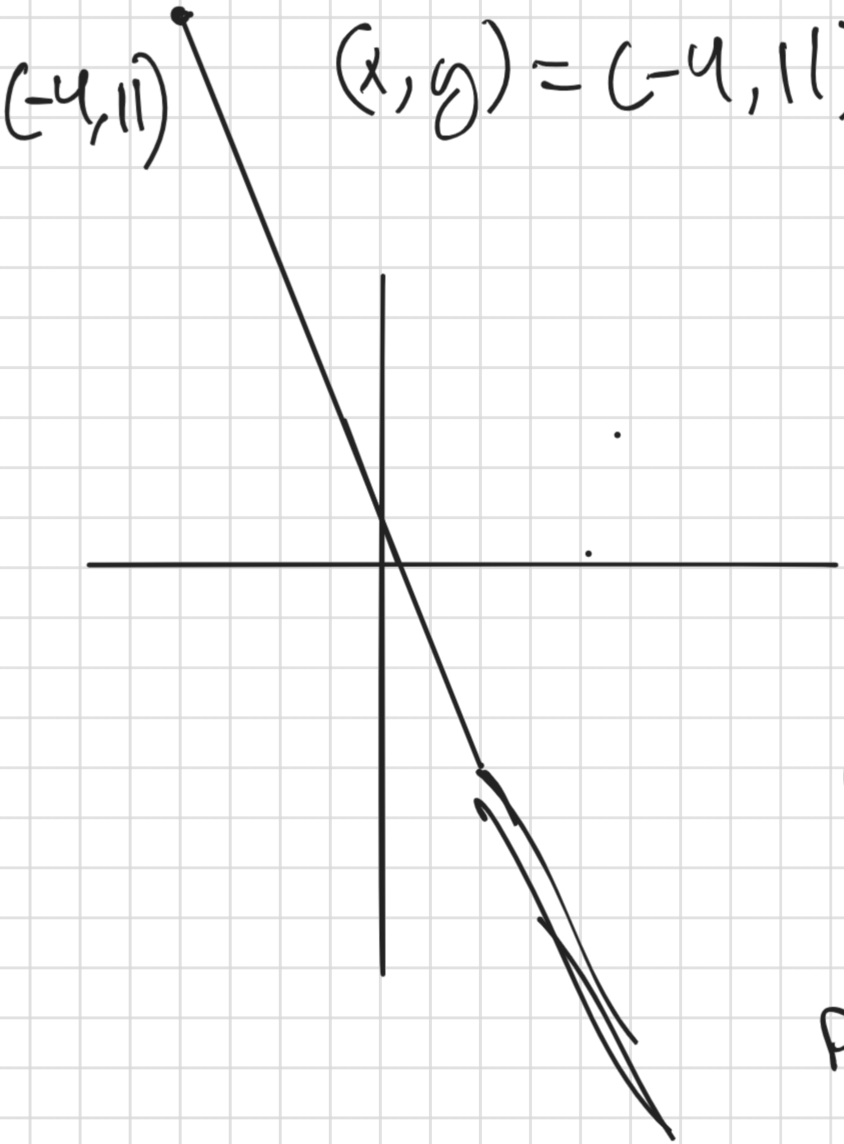
$60(-4) + 22(11) = 2$ $\quad ax + by = \gcd(a,b)$

$(x,y) = (-4,11)$ solves $60x + 22y = 2$



Real solutions are a line

Rewrite as

$$30x + 11y = 1$$

$$y = \frac{-30}{11}x + \frac{1}{11}$$

Changing x to $x + t$ add $\frac{-30}{11}t$ to y.

Points on this line are

$$(x_0, y_0) + (t, mt)$$

$$(-4, 11) + \left(t, \frac{-30}{11}t\right)$$

All real solutions. To get integers, t and $\frac{-30}{11}t$ have to be integers.

So $t = 11s$ for s an integer.

$$(-4, 11) + (11s, -30s)$$
$$= (-4 + 11s, 11 - 30s) \text{ for } s \in \mathbb{Z}.$$