Hello, Is this legible? Week 2, Th (1/20)

Continued Fractions + solving the gcd equation.

\[ a, b \mid ax + by = \gcd(a, b). \]

Quiz something like

Express \([1; 2, 3, 4] \) as a reduced fraction.

\([1; 2, 3, 4] = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}} \]

Continued Fraction.

\[ = 1 + \frac{1}{2 + \frac{1}{\frac{17}{4}}} \]

\[ = 1 + \frac{1}{2 + \frac{4}{13}} \]

\[ = \frac{\alpha}{\beta} \]

Answer: \[ \frac{17}{4} \]
Exercise 2 on worksheet

Recall: A real number \( \alpha \) is rational \( \iff \) the continued fraction expansion of \( \alpha \) is finite, \( \alpha = [a_0, a_1, \ldots, a_m] \). Can use this to show a number is rational or irrational.

Example: \( \sqrt{2} = [1; 2, 2, 2, 2, \ldots] \) is infinite so \( \sqrt{2} \) is irrational.
Expanded Example: \[ \left( 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ddots} } } \right) \]

How to show \( \sqrt{2} = [1; 2, 2, 2, \ldots] \)

Follow algorithm! (see the lecture or accompanying notes)

1. \[ \sqrt{2} = \ln \sqrt{2} + \left( \sqrt{2} - \ln \sqrt{2} \right) \]
   \[ = 1 + \left( \sqrt{2} - 1 \right) \]
   \[ = 1 + \left( \sqrt{2} - 1 \right) \left( \sqrt{2} + 1 \right) \]
   \[ = \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \]
   \[ \Rightarrow \ln \sqrt{2} = 1 \]

2. \[ \sqrt{2} - 1 = \frac{1}{\sqrt{2} - 1} \]
   \[ = \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \]
   \[ = 2 - \frac{1}{\sqrt{2} + 1} \]
   \[ = 2 - \frac{1}{\sqrt{2} + 1} \]
   \[ = 2 - \frac{1}{\sqrt{2} - 1} \]
   \[ \Rightarrow \sqrt{2} = 2 - \frac{1}{\sqrt{2} - 1} \]

3. \[ \sqrt{2} - 1 = \frac{1}{\sqrt{2} - 1} \]
   \[ = \frac{1}{\sqrt{2} - 1} \]
   \[ = \frac{1}{\sqrt{2} - 1} \]
   \[ = \frac{1}{\sqrt{2} - 1} \]
   \[ = \frac{1}{\sqrt{2} - 1} \]

get same thing! So computation repeats.
Using the table

\[ \alpha = \sqrt{2} \]

\[ \alpha_0 = \sqrt{2} \]

\[ \alpha_i = \lfloor \alpha_i \rfloor \]

\[ \beta_i = \alpha_i - \lfloor \alpha_i \rfloor \]

\[ \alpha_{i+1} = \frac{1}{\beta_i} \]

<table>
<thead>
<tr>
<th>i</th>
<th>( a_i = \lfloor \alpha_i \rfloor )</th>
<th>( \beta_i = \alpha_i - \lfloor \alpha_i \rfloor )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>( \sqrt{2} - 1 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>( \sqrt{2} - 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>3</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

This entry determines the next row completely.
Exercise 1 - (b)
Find all integer solutions of $ax + by = \gcd(a,b)$
(first three parts).

In video - can find one solution by unravelling Euclidean algorithm.

Example $a = 60$, $b = 22$, $\gcd(a,b) = 2$

\[
\begin{align*}
60 &= 2 \cdot 22 + 16 \\
22 &= 1 \cdot 16 + 6 \\
16 &= 2 \cdot 6 + 4 \\
6 &= 1 \cdot 4 + 2 \\
4 &= 2 \cdot 2 + 0
\end{align*}
\]

$a \cdot x + b \cdot y = \gcd(a,b)$.

\[
\begin{align*}
60(-4) + 22(11) &= 2 \\
2 &= 1 \cdot 6 - 1 \cdot 4 \\
2 &= 1 \cdot 6 - 1(1 \cdot 16 - 2 \cdot 6) \\
2 &= 3 \cdot 6 - 1 \cdot 16 \\
2 &= 3 \cdot (1 \cdot 22 - 1 \cdot 16) - 1 \cdot 16 \\
2 &= 3 \cdot 22 + (-4) \cdot 16 \\
2 &= 3 \cdot 22 + (-4)(60 - 2 \cdot 22) \\
2 &= 11 \cdot 22 + (-4) \cdot 60
\end{align*}
\]
\( 60(-y) + 22(11) = 2 \)

\((x, y) = (-y, 11)\) solves

\(60x + 22y = 2\)

Real solutions are a line

Rewrite as

\(70x + 11y = 1\)

\(y = -\frac{30}{11}x + \frac{1}{11}\)

Chose \(x \to x + t\)

Add \(-\frac{30}{11}t\) to \(y\).

Points on the line are

\((x_0, y_0) + (t, m + t)\)

\((-y, 11) + (++\frac{30}{11}t\)

All real solutions to get integers, \(t\) and \(-\frac{30}{11}t\) have to be integers.
So \( t = 11s \) for \( s \in \mathbb{Z} \).

\[
(-4, 11) + (11s, -30s) = (-4 + s, 11 - 30s) \quad \text{for } s \in \mathbb{Z}.
\]