Some of these exercises can be found in Savin - Chapter 9.

Exercise 1 (required). Pentagonal numbers. The $n$th pentagonal number $P_n$ is the number of dots in a regular pentagon whose sides have $n$ dots. For example, $P_1 = 1$, $P_2 = 5$.

(1) Draw the dot diagrams to compute $P_n$ for $n \leq 5$.

(2) Find a simple formula for $P_{n+1} - P_n$.

(3) Use this formula and the identity $1 + \ldots + (n-1) = \frac{(n-1)n}{2}$ to find a simple formula for $P_n$.

(4) A square-pentagonal number is a number that is both square and pentagonal. Make a substitution in the equation for finding square-pentagonal numbers

$$m^2 = P_n$$

to turn it into the Pell equation $x^2 - 6y^2 = 1$.

(5) Use that $1^2 = P_1$ to find a solution to $x^2 - 6y^2 = 1$, and verify it is the smallest solution (i.e. the solution with the smallest value of $x$ or equivalently with the smallest value of $y$).

(6) Using $N(x + \sqrt{6}y) = x^2 - 6y^2$ and multiplicativity of the norm, we can compute new solutions to $x^2 - 6y^2 = 1$ by raising to powers $(x + y\sqrt{6})^k$ for $(x, y)$ as in part (5). Not all of these will give rise to square-pentagonal numbers, because solving for $m$ and $n$ in terms of $x$ and $y$ may not always give integers. Use the fact that

$$N(x + \sqrt{6}y) = x^2 - 6y^2$$

to produce new solutions from your first solution by raising $(x + y\sqrt{6})$ to powers until you find a non-trivial square-pentagonal number.

(7) (Optional – not to turn in) Find a triangular-pentagonal number larger than 1.
Exercise 2 (required). Two principles
(1) Use Dirichlet’s principle with \( n = 10 \) to find a rational approximation of \( \pi \).

(2) Use the pigeonhole principle to show that there is a multiple of 691 whose digits are 0s and 1s. Hint: consider the sequence 1, 11, 111, . . . .

Exercise 3. Units in \( \mathbb{Z}[\sqrt{D}] \). Let \( D \) be a positive squarefree integer.
(1) Show that \( \alpha = a + b\sqrt{D} \in \mathbb{Z}[\sqrt{D}] \) is a unit if and only if
\[
N(\alpha) = (a + b\sqrt{D})(a - b\sqrt{D}) = a^2 - b^2D = \pm 1
\]

(2) If \( N(\alpha) = 1 \), then \( a \) and \( b \) are a solution to Pell’s equation \( a^2 - Db^2 = 1 \). Find an example of a \( D \) and unit \( \alpha \in \mathbb{Z}[\sqrt{D}] \) such that \( N(\alpha) = -1 \).

(3) For which primes \( p \) is there is a unit \( \alpha \in \mathbb{Z}[\sqrt{p}] \) with \( N(\alpha) = -1 \)?

(4) Find a unit \( \gamma \in \mathbb{Z}[\sqrt{3}] \) such that all other units are of the form \( \pm \gamma^k \).

Exercise 4. Continued fractions and Pell’s equations. It turns out that we can use continued fractions to solve Pell’s equations! This is related to a fact we observed much earlier (that continued fractions are periodic for quadratic numbers). To learn about this connection, read Savin - Chapter 9, Section 5 and then do the exercises in that section.