4400-001 - SPRING 2022 - WEEK 13 (4/12, 4/14)
SHAPE NUMBERS, PELL'S EQUATIONS, AND RATIONAL APPROXIMATION

Some of these exercises can be found in Savin - Chapter 9.

Exercise 1 (required). Pentagonal numbers. The nth pentagonal number P_n is the number of dots in a regular pentagon whose sides have n dots. For example, $P_1 = 1$, $P_2 = 5$.

(1) Draw the dot diagrams to compute P_n for $n \leq 5$.

(2) Find a simple formula for $P_{n+1} - P_n$.

(3) Use this formula and the identity $1 + \ldots + (n-1) = \frac{(n-1)n}{2}$ to find a simple formula for P_n.

(4) A square-pentagonal number is a number that is both square and pentagonal. Make a substitution in the equation for finding square-pentagonal numbers

$$m^2 = P_n$$

to turn it into the Pell equation $x^2 - 6y^2 = 1$.

(5) Use that $1^2 = P_1$ to find a solution to $x^2 - 6y^2 = 1$, and verify it is the smallest solution (i.e. the solution with the smallest value of x or equivalently with the smallest value of y).

(6) Using $N(x + \sqrt{6}y) = x^2 - 6y^2$ and multiplicativity of the norm, we can compute new solutions to $x^2 - 6y^2 = 1$ by raising to powers $(x + y\sqrt{6})^k$ for (x, y) as in part (5). Not all of these will give rise to square-pentagonal numbers, because solving for m and n in terms of x and y may not always give integers. Use the fact that

$$N(x + \sqrt{6}y) = x^2 - 6y^2$$

to produce new solutions from your first solution by raising $(x + y\sqrt{6})$ to powers until you find a non-trivial square-pentagonal number.

(7) **(Optional – not to turn in)** Find a triangular-pentagonal number larger than 1.

Exercise 2 (required). Two principles

(1) Use Dirichlet's principle with $n = 10$ to find a rational approximation of π.

(2) Use the pigeonhole principle to show that there is a multiple of 691 whose digits are 0s and 1s. *Hint: consider the sequence 1, 11, 111, ...*

Exercise 3. Units in $\mathbb{Z}[\sqrt{D}]$. Let D be a positive squarefree integer.

(1) Show that $\alpha = a + b\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$ is a unit if and only if

$$N(\alpha) = (a + b\sqrt{D})(a - b\sqrt{D}) = a^2 - b^2 D = \pm 1$$

(2) If $N(\alpha) = 1$, then a and b are a solution to Pell's equation $a^2 - Db^2 = 1$. Find an example of a D and unit $\alpha \in \mathbb{Z}[\sqrt{D}]$ such that $N(\alpha) = -1$.

(3) For which primes p is there is a unit $\alpha \in \mathbb{Z}[\sqrt{p}]$ with $N(\alpha) = -1$?

(4) Find a unit $\gamma \in \mathbb{Z}[\sqrt{3}]$ such that all other units are of the form $\pm\gamma^k$.

Exercise 4. Continued fractions and Pell's equations. It turns out that we can use continued fractions to solve Pell's equations! This is related to a fact we observed much earlier (that continued fractions are periodic for quadratic numbers). To learn about this connection, read Savin - Chapter 9, Section 5 and then do the exercises in that section.