Math 4400
Week 13 - Tuesday
Shape numbers and Pell's equation

<table>
<thead>
<tr>
<th>n</th>
<th>$S_n$</th>
<th>n</th>
<th>$T_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

\[ T_n = \frac{n(n+1)}{2} \]
\[ S_n = n^2 = 1 + 3 + 5 + 7 + \ldots + (2n-1) = \frac{n(n+1)}{2} \]

\[
\begin{array}{cccccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 S_n & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 \\
 T_n & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 \\
\end{array}
\]

1 and 36 are both square and triangular

**Question:** Which integers \( n \) are both square and triangular? (square-triangular)

To answer, need to find the solutions to \( T_m = S_n \)

i.e.

\[ m(m+1)/2 = n^2 \]

\[ \underline{\xi \leq 8} \]

Rewrite:

\[ 4m^2 + 4m = 8n^2 \]

\[ (2m+1)^2 - 1 = 8n^2 \]

Substitute: \( x = 2m+1, \ y = 2n \)
\[ x^2 - 1 = 2y^2 \]
\[ \uparrow \]
\[ x^2 - 2y^2 = 1. \]

\[(m, n) \text{ s.t. } T_m = S_n \rightarrow (x, y) \text{ s.t. } x^2 - 2y^2 = 1\]
\[ (1, 1) \quad 1 = 1 \rightarrow (3, 2) \quad 3^2 - 2.2^2 = 1 \]
\[ (8, 6) \quad \frac{8}{36} = \frac{6}{36} \rightarrow (17, 12) \quad 17^2 - 2.12^2 = 1 \quad \text{where } 289 - 288 = 1. \]

Where it gets cool:

\[ x^2 - 2y^2 = N(x + y\sqrt{2}) = (x + y\sqrt{2})(x - y\sqrt{2}). \]

Note \( N(ab) = N(a)N(b) \) for \( a, b \in \mathbb{Z}[\sqrt{2}] \).

Example: \( N(3 + 2\sqrt{2}) = 3^2 - 2.2^2 = 1. \)

so for any \( k \),

\[ N((3 + 2\sqrt{2})^k) = N((3 + 2\sqrt{2})^k) = 1 \quad k = 1. \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>((3 + 2\sqrt{2})^k)</th>
<th>((x, y))</th>
<th>((m, n))</th>
<th>(T_m = S_n)</th>
<th>square-triangle #</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(3 + 2\sqrt{2})</td>
<td>((3, 2))</td>
<td>((1, 1))</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>(17 + 12\sqrt{2})</td>
<td>((17, 12))</td>
<td>((8, 6))</td>
<td>36</td>
<td></td>
</tr>
</tbody>
</table>
Amazing fact: this gives every integer solution to \( x^2 - 2y^2 = 1 \)

\[
T_m = S_n
data - triangular #
\]

Pell's equations: \( D > 0 \) square free, integer solve \( x^2 - Dy^2 = 1 \)
Same strategy works (use \( \mathbb{Z}[\sqrt{D}] \)).

Next time: Why is there always a solution?