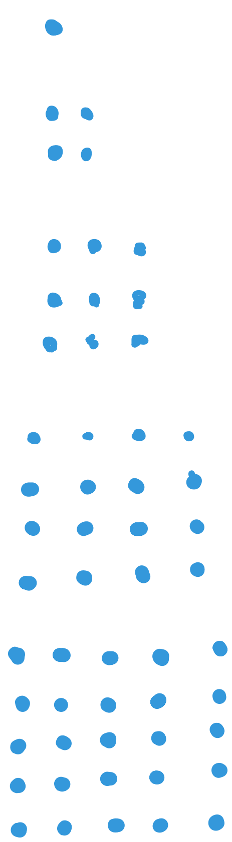
Math 4400

Week 13 - Tuesday

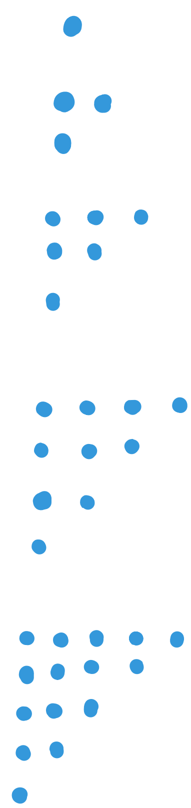
Shape numbers and Pell's equation

Square numbers

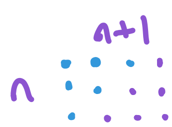


n	S_n
1	1
2	4
3	9
4	16
5	25

Triangular numbers



T_n
1
3
6
10
15



$S_n = n^2$

$= 1+3+5+7+\dots+(2n-1)$

$T_n = 1+2+3+\dots+n.$

$= \frac{n(n+1)}{2}$

n	1	2	3	4	5	6	7	8	9
S_n	1	4	9	16	25	36	49	64	81
T_n	1	3	6	10	15	21	28	36	45

1 and 36 are both square and triangular

Question: Which integers n are both square and triangular?
(square-triangular)

To answer, need to find the solutions to

$$T_m = S_n$$

i.e.

$$m(m+1)/2 = n^2$$

$\cdot 8$

Rewrite: $$4m^2 + 4m = 8n^2$$
$$(2m+1)^2 - 1 = 8n^2$$

Substitute: $x = 2m+1 \quad y = 2n$

$$x^2 - 1 = 2y^2$$
$$\Updownarrow$$
$$x^2 - 2y^2 = 1.$$

(m, n) s.t. $T_m = S_n$ $\rightsquigarrow$ (x, y) s.t. $x^2 - 2y^2 = 1$

$(1, 1)$ $\quad 1 = 1$ $\rightsquigarrow$ $(3, 2)$ $\quad 3^2 - 2 \cdot 2^2 = 1$

$(8, 6)$ $\quad \underset{T_8}{36} = \underset{S_6}{36}$ $\rightsquigarrow$ $(17, 12)$ $\quad 17^2 - 2 \cdot 12^2 = 1$

$289 - 288 = 1.$

Where it gets cool:

$$x^2 - 2y^2 = N(x + y\sqrt{2}) = (x + y\sqrt{2})(x - y\sqrt{2}).$$

Note $N(ab) = N(a)N(b)$ for $a, b \in \mathbb{Z}[\sqrt{2}]$

Example: $N(3 + 2\sqrt{2}) = 3^2 - 2 \cdot (2)^2 = 1.$

so for any K,

$$N((3 + 2\sqrt{2})^K) = N(3 + 2\sqrt{2})^K = 1^K = 1.$$

K	$(3 + 2\sqrt{2})^K$	(x, y) $x^2 - 2y^2 = 1$	(m, n) $T_m = S_n$	square-triangular #
1	$3 + 2\sqrt{2}$	$(3, 2)$	$(1, 1)$	1
2	$17 + 12\sqrt{2}$	$(17, 12)$	$(8, 6)$	36

3	$99 + 70\sqrt{2}$	$(99, 70)$	$(49, 35)$	1225
$\vdots$				

$\frac{49 \cdot 50}{2} \;//\; //\; 35^2$

Amazing fact: this gives <u>every</u>

integer solution to $x^2 - 2y^2 = 1$

$\Updownarrow$

$T_m = S_n$

$\Updownarrow$

square-triangular #

Pell's equations: $D > 0$ square free, integer

solve $x^2 - Dy^2 = 1$

Same strategy works (use $\mathbb{Z}[\sqrt{D}]$).

Next time: Why is there always a solution?