In the video: square triangular numbers

\[ \sum \]

Pell's equation \( x^2 - 2y^2 = 1 \)

(Pell's equations: \( x^2 - Dy^2 = 1 \) \( D \) squarefree integer)

\[ x^2 - Dy^2 = N(x + y\sqrt{D}) \]

Key thing: \( N(ab) = N(a)N(b) \)
\( a, b \in \mathbb{Z}[\sqrt{D}] \).

If \( z = x + y\sqrt{D} \) satisfies \( N(z) = 1 \)

Then \( N(z^n) = N(z)^n = 1 \).
Exercise 1: Square-pentagonal #s

\[ n = 1 \quad 2 \quad 3 \]

Dots: \[ \cdot \quad \cdot \quad \cdot \quad \cdot \]

\[ p_n = 1 \quad 5 \quad 12 \]

Have at it!

Hint (2): \[ p_{n+1} - p_n = \] 
\# of dots you add to go from \( n \)th to the \( n+1 \)st.

\[ p_{n+1} - p_n = 3n + 1 \] 
filling in the vertex, propagating each side not on the left or right.
Hint (3): \[ p_n = 1 + q + r + \ldots + (3(n-1)+1). \]

\[ = p_0 + (p_1 - p_0) + (p_2 - p_1) + \ldots + (p_n - p_{n-1}) \]

\[ = 0 + \sum_{k=0}^{n-1} (3k+1) \]

\[ = \sum_{k=0}^{n-1} (3k+1) \quad \text{by part 2}, \]

\[ = \sum_{k=0}^{n-1} (3k+1) \quad \text{Now need to simplify this.} \]

(If you think about \( T_{n-1} = 1 + \ldots + n-1 = \frac{(n-1)n}{2} \),

there is also a nice geometric way to get a formula.)

Hint (4): Complete the square, don't be afraid of rational fits.
\[(4)\]

\[
m^2 = \frac{3(n-1)(n)}{2} + n
\]

\[
m^2 = \frac{3n^2 - 3n + 2n}{2} = \frac{3n^2 - n}{2}
\]

\[
\frac{2}{3} m^2 = n^2 - \frac{n}{3}
\]

\[
\frac{2}{3} m^2 = \left(\frac{n - \frac{1}{6}}{6}\right)^2 - \frac{1}{6^2}
\]

\[
24m^2 = (6n - 1)^2 - 1
\]

\[
1 = (6n - 1)^2 - 24m^2
\]
(5) "Verify it's the smallest solution of all solutions with positive integer $x$ and $y$ values, the one with the smallest $x$ (or the smallest $y$) is same because $X^2 = 1 + 6y^2$.

Can check a given solution $x_0, y_0$ is smallest by plugging in all $1 \leq x < x_0$ or $1 \leq y < y_0$ and seeing they don't give a solution.

e.g. $y = 1$ gives $x^2 = 1 + 6 \cdot 1^2 = 7$ but 7 is not a square!"