In the video: square-triangular numbers

$\Updownarrow$

Pell's equation $x^2 - 2y^2 = 1$

(Pell's equations: $x^2 - Dy^2 = 1$
D squarefree integer).

$$x^2 - Dy^2 = N(x + y\sqrt{D})$$

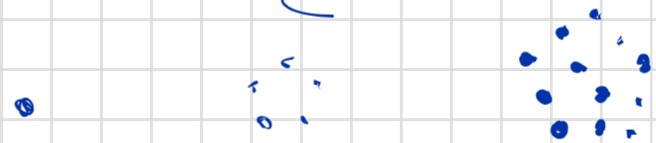
Key thing: $N(ab) = N(a)N(b)$
$a, b \in \mathbb{Z}[\sqrt{D}]$.

If $z = x + \sqrt{D}\, y$ satisfies $N(z) = 1$

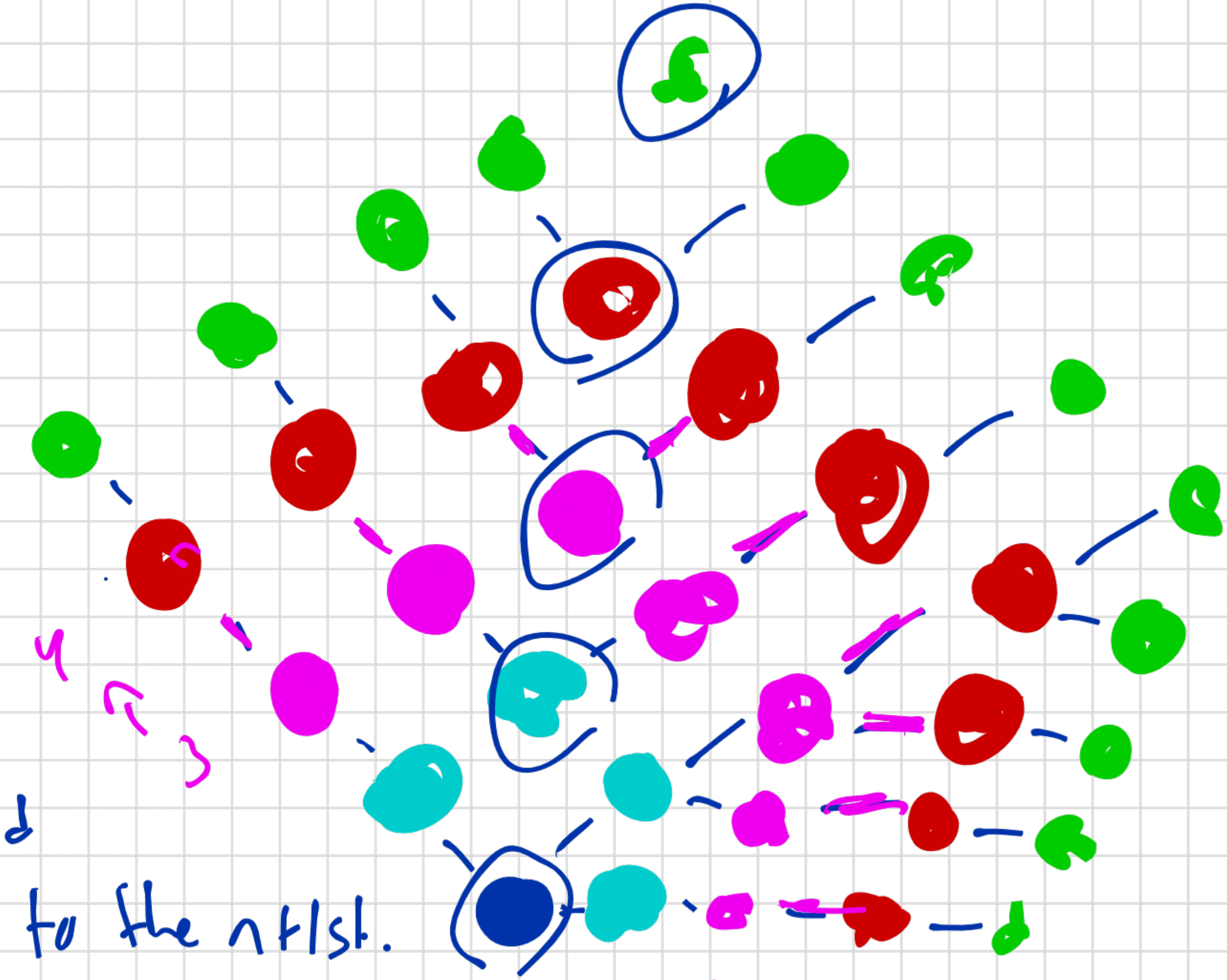
Then $N(z^k) = N(z)^k = 1$.

Exercise 1: square-pentagonal #s

$n =$	1	2	3
Dots			
$P_n =$	1	5	12



Have at it!



Hint (2): $P_{n+1} - P_n =$ # of dots you add to go from nth to the n+1st.

$$P_{n+1} - P_n = 3n + 1$$

filling in the vertex, propagating each side out on the left or bottom.

Hint (3): $P_n = 1 + 4 + 7 + \ldots + (3(n-1)+1)$.

$$= P_0 + (P_1 - P_0) + (P_2 - P_1) + \ldots + (P_n - P_{n-1})$$

$$= 0 + \underbrace{\sum_{k=0}^{n-1} (3k+1)}_{\text{by part 2.}}$$

$$= \sum_{k=0}^{n-1} (3k+1)$$

← Now need to simplify this.

(If you think about $T_{n-1} = 1 + \ldots + n-1 = \frac{(n-1)n}{2}$, there is also a nice geometric way to get a formula).

Hint (4): Complete the square, don't be afraid of rational #s.

(4)

$$m^2 = \frac{3(n-1)(n)}{2} + n$$

$$m^2 = \frac{3n^2 - 3n + 2n}{2}$$

$$m^2 = \frac{3n^2 - n}{2} = \frac{3}{2}n^2 - \frac{n}{2}.$$

$\cdot \frac{2}{3} \downarrow$

$$\frac{2}{3}m^2 = n^2 - \frac{n}{3}$$

$$\frac{2}{3}m^2 = \left(n - \frac{1}{6}\right)^2 - \frac{1}{6^2}$$

$\downarrow \cdot 6^2$

$$24m^2 = (6n-1)^2 - 1$$

$$1 = (6n-1)^2 - 24m^2$$

$x = 6n - 1$

$y = 2m$

$$\boxed{1} = x^2 - 6y^2.$$

(5) "Verify it's the smallest solution"

of all solutions with positive integer x and y values, the one with the smallest x (or the smallest y) } Same because $x^2 = 1 + 6y^2$.

Can check a given solution x_0, y_0 is smallest by plugging in all $1 \le x < x_0$ or $1 \le y < y_0$ and seeing they don't give a solution.

e.g. $y=1$ gives $x^2 = 1 + 6 \cdot 1^2 = 7$ but 7 is not a square!