Last time: once we have one solution \((x_1, y_1)\) to \(x^2 - Dy^2 = 1, \ D \text{ square free}\), can produce many more by:

\[(x_1 + y_1 \sqrt{D})^k = x_k + y_k \sqrt{D}\]

\[NC(-k) = N(-k)^k = 1.\]

Fact: If \((x_1, y_1)\) is the smallest positive integer solution (i.e. \(x_1\), smaller than \(x\) coordinate of any other or equivalently \(y_1\), smaller than \(y\) coordinate of any other),

\[x^2 = 1 + Dy^2\]
then all positive integer solutions are as above.

Example: Last time, $D = 2$, the smallest positive integer solution to
$x^2 - 2y^2 = 1$
was $(x_1, y_1) = (3, 2)$
Taking $(x_1 + \sqrt{2}y_1)^k$ gave all other solutions, thus all square-triangular #s.

To justify fact, we divide by $x_1 + \sqrt{2}y_1$:

\[
\frac{(x + y\sqrt{2})}{(x_1 + y_1\sqrt{2})} = \frac{(x + y\sqrt{2})(x_1 - y_1\sqrt{2})}{x_1^2 - 2y_1^2} = x_1 - \sqrt{2}y_1 + (x_1y_1 - xy_1\sqrt{2}) \sqrt{2}
\]

Need to show these are positive — see Theorem 43 on p. 129.

Just keep dividing!

More interesting — to have a smallest solution, first need to know there is at least one positive integer solution! to Pell's equation $x^2 - y^2D = 1$. 
Key observation: If \( x^2 - Dy^2 = 1 \) w/ \( x, y > 0 \)
\((x + \sqrt{D}y)(x - \sqrt{D}y) = 1\)
\(x - \sqrt{D}y = \frac{1}{x + \sqrt{D}y} < \frac{1}{y}\).

So \( |x - \sqrt{D}y| < \frac{1}{y} \)

\( \frac{1}{y} \) says \( \frac{x}{y} \) is fraction that approximates \( \sqrt{D} \) well.

Example: \( 99^2 - 2.70^2 = 1 \)
\( \frac{99}{70} = 1.4142 \ldots \quad \sqrt{2} = 1.41421 \ldots \)
agree up to first 4 decimal places

Idea: Reverse this - use existence of good rational approximations to find a solution.

Pigeonhole principle: If I put \( n+1 \) pigeons into \( n \) pigeonholes, then
one pigeonhole holds at least 2 pigeons.
Also: If infinitely many pigeons in finitely many pigeonholes, then one pigeonhole holds infinitely many pigeons.

Dirichlet approximation principle: If $\alpha > 0$ is irrational, then there are infinitely many pairs $(x, y)$ of positive integers such that

$$\left| \frac{x}{y} - \alpha \right| < \frac{1}{y^2}.$$  

Proof: Let $n > 0$. Subdivide $(0, 1)$ into $n$ intervals of length $\frac{1}{n}$.

$n$ Pigeonholes: $\left[0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), \ldots, \left[\frac{n-1}{n}, 1\right)$. 

$n+1$ Pigeons: $\{0, \alpha\}, \{1, \alpha\}, \ldots, \{n, \alpha\}$

$\{\beta\} = \beta - \lfloor \beta \rfloor$ is the fractional part.

Pigeonhole principle $\Rightarrow$ there is $0 \leq k \leq n-1$ and $0 \leq i < j \leq n$ s.t.

$\{i\alpha\}$ and $\{j\alpha\}$ are both in $\left[\frac{k}{n}, \frac{k+1}{n}\right)$.

So $(j-i)\alpha = j\alpha - i\alpha = \lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor + \{j\alpha\} - \{i\alpha\}$

$$\alpha = \frac{\lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor}{j-i} + \frac{\{j\alpha\} - \{i\alpha\}}{j-i}$$

$$\alpha = \frac{x}{y} + \varepsilon \quad y = j-i$$
\[ |e| = \left| \sum_{j \neq i} \frac{(j \cdot x_j - i \cdot x_i)}{y_{j-i}^2} \right| < \frac{1}{y^2} = \frac{1}{\sqrt{y}} < \frac{1}{\sqrt{n}} \] 

Existence of solutions to Pell's equation:

For any \( n \), positive integers \( x \leq \sqrt{nD} \), \( y \leq n \) \( s.t. \) \( |x - y\sqrt{D}| < \frac{1}{n} \) (see proof of Dirichlet).

Then \( N(x + y\sqrt{D}) = (x + y\sqrt{D})(x - y\sqrt{D}) < \frac{2n\sqrt{D}}{\sqrt{n}} = 2\sqrt{D} \)

Pigeonhole (again!) \( \Rightarrow \exists \ m < 2\sqrt{D} \) \( s.t. \)
\[ N(x + y\sqrt{D}) = m \] has infinitely many positive integer solutions \((x,y)\).

For any two,
\[ N\left(\frac{x_1 + y_1\sqrt{D}}{x_2 + y_2\sqrt{D}}\right) = \frac{N(x_1 + y_1\sqrt{D})}{N(x_2 + y_2\sqrt{D})} = \frac{m}{m} = 1. \]

\[ \frac{x_1 + y_1\sqrt{D}}{x_2 + y_2\sqrt{D}} = \frac{(x_1 + y_1\sqrt{D})(x_2 - y_2\sqrt{D})}{m} = \frac{x_1x_2 - y_1y_2D + (y_1x_2 - x_1y_2)\sqrt{D}}{m}. \]

\[ \text{want integers.} \]
Pigeonhole principle one more time:
can assume \(x_1 \equiv x_2 \mod m\) and \(y_1 \equiv y_2 \mod m\).
(Pigeonholes - \((a, b)\) in \((\mathbb{Z}/m\mathbb{Z})^2\).
\[x \equiv a, \quad y \equiv b\]

Then \(x_1 x_2 - y_1 y_2 D \equiv x_1^2 - y_1^2 D \equiv m \equiv 0 \mod m\)
\(y_1 x_2 - x_1 y_2 \equiv y_1 x_1 - x_1 y_1 \equiv 0 \mod m\). 
\(\checkmark\)