Sums of 2 squares.

Method of descent:

If $p \equiv 1 \pmod 4$ is prime

want to write $p = a^2 + b^2$.

<u>Step 1</u>: Find $a^2 + b^2 = m p$.

<u>Step 2</u>: Descent – an algorithm to find $(a')^2 + (b')^2 = m' p$

$m' \le \frac{m}{2}$.

Repeat until you find $p = x^2 + y^2$.

Exercise 1 - (2) - (3): follow method of descent given a starting equation.

To do this:

- $a^2 + b^2 = mp$
- $u \equiv a \mod m \qquad v \equiv b \mod m$

$$-\frac{m}{2} < u, v \leq \frac{m}{2}.$$

- $(u - iv)(a + bi) = c + di$

$$N(c+di) = c^2 + d^2 = N(u-iv)N(a+bi) = m^2 r p$$

$r \leq \frac{m}{2}$.

$a' = \frac{c}{m}$ $\qquad b' = \frac{d}{m}$

$N(a' + b'i) = (a')^2 + (b')^2 = rp$, a', b' are both integers.

Do Ex 1, parts (2) - (3) now! $\qquad$ (2) $11^2 + 1^2 = 2 \cdot 61$

Example: $\underset{a}{11}^2 + \underset{b}{1}^2 = \underset{m}{2} \cdot \underset{p}{61}$

So take $u \equiv 11 \bmod 2$ $\qquad v \equiv 1 \bmod 2$

$-1 < u, v \leq 1 \quad (1 = \frac{m}{2})$

$\Rightarrow u = 1 \qquad v = 1$

$(1-i)(11+i) = 12 - 10i$

$$\frac{12-10i}{2} = 6-5i.$$

$$6^2+(-5)^2 = 61 \quad \checkmark.$$

$$\overset{a}{84}^2 + \overset{b}{89}^2 = \overset{m}{17} \cdot \overset{p}{881}$$

$$u \equiv 84 \text{ mod } 17 \qquad v \equiv 89 \text{ mod } 17$$

$$-\frac{m}{2} < u, v \le \frac{m}{2} \qquad \frac{m}{2} = 8.5.$$

$$\Rightarrow \quad u = -1 \qquad v = 4 \qquad (5 \cdot 17 = 85).$$

$$(\overset{u}{-1} \overset{-\,vi}{-\,4i})(\overset{a}{84} \overset{+\,bi}{+\,89i}) = (-84+4\cdot 89) + (-89-4\cdot 84)i$$

$$= 272 + (-425)i$$

$\overline{17}$ gives $16-25i$

$16^2 + (-25)^2 = 881$ ✓

Announcement: Next week (12, 14)
is the last material for the final.

· The final will be cumulative

· On April 26th review in class (April 28th)

Before then, I'll post template/info like for midterm.

April 19, 21 we'll cover an optional topic (elliptic curve cryptography)

Quizzes and assignment ~ Bonus points.

To find $a^2+b^2=mp$.

Suffices to find a s.t. $a^2 \equiv -1 \mod p$

then $a^2+1^2 \equiv 0 \mod p$ $(a^2+1^2=mp)$.

So can take $b=1$.

Since $p \equiv 1 \mod 4$, we know there exists

a square root of $-1 \mod p$;

how to find it?

① Guess till you find it.

② If you know primitive root $g \mod p$.

Then FACT: $g^{\frac{p-1}{4}}$ is a square root of -1 mod p.

1.(4) – justify this fact.

1.(5) – use this in a specific case to find $a^2+b^2=np$.
($a^2+1^2\equiv np$)

(5)-(a) 5^{18} ~ compute it mod 73.
the whole thing

i.e, take a to be the smallest integer $\equiv 5^{18}$ mod 73.

made with ZITEBOARD

(4) In a field, to check

$\alpha = -1$, it suffices to check that $\alpha^2 = 1$ and $\alpha \neq 1$.

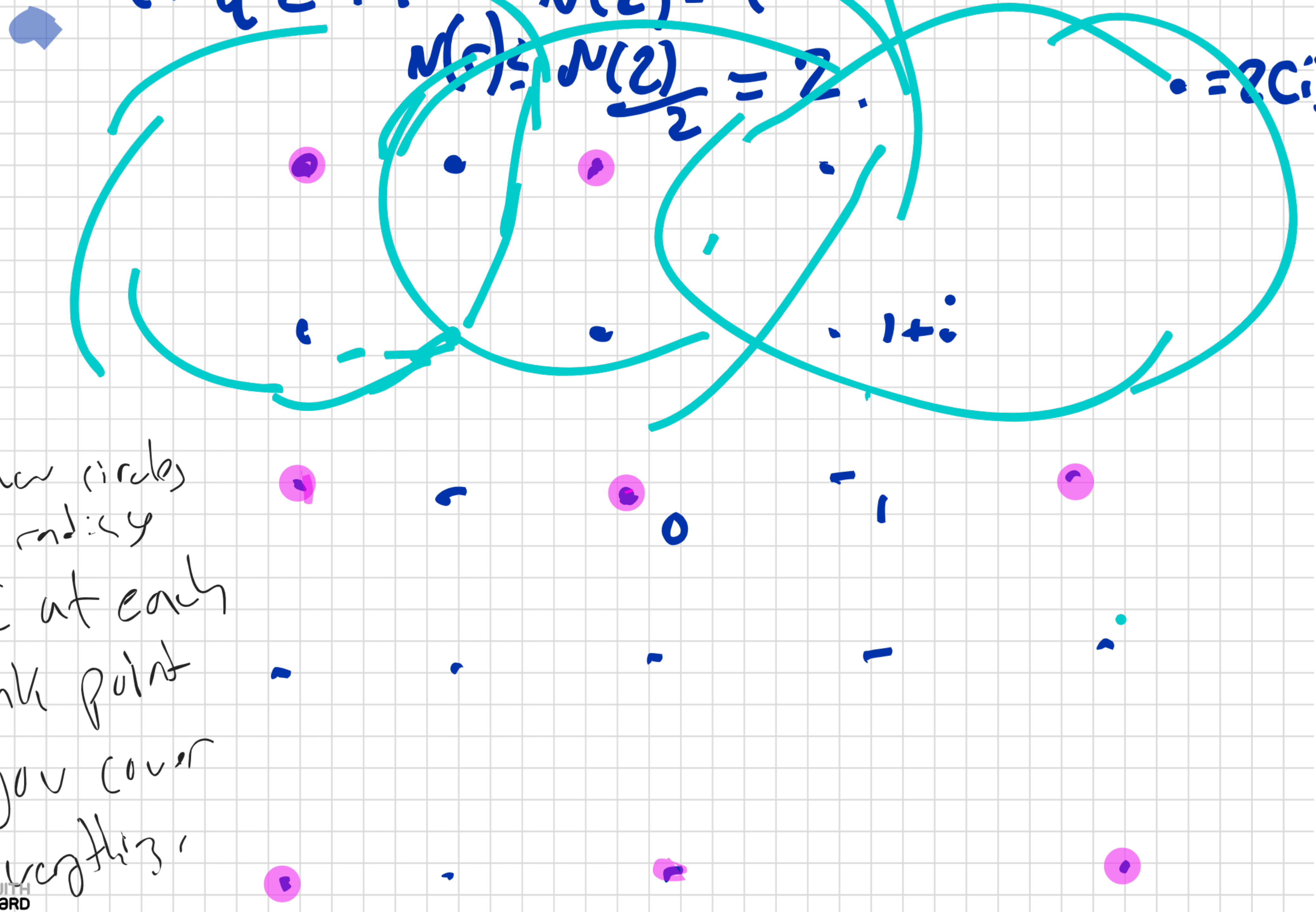
Apply to $\alpha = a^2$. $a = g^{\frac{p-1}{4}}$.

To check $a^2 \neq 1$ need to use that g is a <u>primitive root</u>.

$\alpha = q \cdot 2 + r$

$N(2) = 4$

$N(r) \leq \frac{N(2)}{2} = 2$



Draw circles of radius $\sqrt{2}$ at each pink point you cover everything!

This doesn't work in $\mathbb{Z}[\sqrt{-5}]$.

$p \equiv 1 \bmod 4$.

Suppose $a^2+b^2 = mp$ Suppose ~~$p \nmid m$~~.

arrange by shifting a, b mod p.

$N(a+bi) = mp$.

$(a+bi)(a-bi) = mp$.

$p = (x+yi)(x-yi)$

$x^2+y^2 = p$.

So $x+yi$ (resp. $x-yi$) has to be factor of one of the LHS.