4400-001 - SPRING 2022 - WEEK 1 (1/11, 1/13)
WAIT, WHAT'S NUMBER THEORY?

Exercise 1. (Required) Use the *sieve of Erastothenes* to find all prime numbers between 1 and 100. The sieve will be described in class on Thursday; alternatively, look it up on Wikipedia!

Exercise 2. (Required) Use the unique factorization of natural numbers (as stated in the video lecture; see the posted PDF of the lecture notes) to explain why $\sqrt{2}$ is not a rational number.

Exercise 3.

(1) Which prime numbers < 100 can be expressed as a sum of two squares? Based on this, can you make a conjecture about which prime numbers will be expressible in this way?
(2) What if we allow four squares?

Exercise 4.

(1) Let $r_k(n)$ denote the number of ways to express a natural number n as a sum of k squares (of integers), i.e. the number of integer solutions to
$$a_1^2 + a_2^2 + \ldots + a_k^2 = n.$$
Don't forget the a_i are allowed to be negative numbers or zero!
(2) Compute $r_2(n)$ for some small values of n.
(3) Compute $r_4(n)$ for some small values of n.
(4) Compute $r_8(n)$ for some small values of n.
(5) Compare your results for $k = 4, 8$ with the following formulas due to Jacobi
$$r_4(n) = 8 \sum_{d|n, 4\nmid d} d$$
$$r_8(n) = 16 \sum_{d|n} (-1)^{n+d} d^3$$
In this class we will eventually be able to explain the formula for $r_4(n)$, as well as a simple formula for $r_2(n)$ that we aren't able to state without introducing some more notation. If you have taken a course in complex analysis, then there is a beautiful geometric proof of the formula for $r_8(n)$ that is within reach, say, for a semester-long independent study project: see the last chapter of Serre's "A Course in Arithmetic."