Fields: Commutative ring s.t. non-zero elements are invertible.

$\implies$ Ring whose only idel is $0$ and $R$.

Examples: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_p$, $\mathbb{F}_p^n$, $\mathbb{Z}/p\mathbb{Z}$

$\text{Frac}\, \mathbb{Z}$.

$C(t) = \text{"field of rational functions in one variable over } C"$

$\left\{ \frac{f(t)}{g(t)} \mid f(t), g(t) \text{ polynomials.} \right\}$

$F(t_1, t_2, t_3, \ldots, t_n) = \text{Frac } F[t_1, \ldots, t_n]$

$\text{Frac } F(t_1, t_2, \ldots) = \text{Frac } F[t_1, t_2, \ldots]$

Weird: $\mathbb{A}(t_1; 1 \in S) \subseteq \mathbb{C}$.

$\mathbb{C}[x, y]/y^2 - (x^3 + 1) \nLeftarrow \text{Integral domain}

\nLeftarrow \text{irreducible.}$

$\text{Frac } \mathbb{C}[x, y]/y^2 - x^3 + 1$ is a field.

$\mathbb{A}^1$ (Weierstrass theory of elliptic functions).

Field of meromorphic functions on $\mathbb{C}$; $\mathbb{C}((z))$.
If \( R \) is a commutative ring and \( \mathfrak{p} \) is a prime ideal then \( \text{Frac}(R/\mathfrak{p}) \) is a field.

If \( \mathfrak{p} \) is maximal then \( R/\mathfrak{p} \) is a field.

\[ \mathbb{Q}_p \rightarrow \text{Complete } \mathbb{Q} \text{ for the } p \text{-adic absolute value.} \]

\( R \rightarrow \text{completed } \mathbb{Q} \text{ for the } p \text{-adic absolute value, } \]
\[ \text{defined by } p \text{-adic absolute value.} \]
\[ \text{Any } \mathbb{Q}_p \text{ is } \mathbb{Q}, \text{ absolute value of measure } \]
\[ \text{distance of measure } \text{by a prime number } p. \]

**Observation:** If \( K \) is a field then there is a unique map \( \phi : \mathbb{Z} \rightarrow K. \)

\[ 1 \mapsto 1 \]
\[ 2i \mapsto 1 + 1 \]
\[ 3i \mapsto 1 + 1 + 1, \]

This map has a kernel, \( \sqrt{\text{Prime ideals of } \mathbb{Z}} \).
\( \phi(p) \) is a prime number. \( \text{In } \phi \cong \mathbb{Z} / \ker \phi \) is an integral domain. (similar of a field).

So \( \ker \phi \) is a prime ideal.

\( \mathbb{Z} \) is a PID so we know what prime ideals are.

Either \( \ker \phi = 0 \) \( \mathbb{Z} \cong K \) by definition of a field.

\( \phi \mathbb{Z} \cong K \)

\( \ker \phi = (p) \)

\( \mathbb{Z}/p\mathbb{Z} \cong K \).

\[ \mathbb{F}_p \]

**Definition/Theorem:** The characteristic of a field \( K \) is the smallest positive \( n \) such that

\[ 1 + 1 + \cdots + 1 = 0 \]

\( n \) times

or 0 if there is no such \( n \).

\( \Rightarrow \)

\( K \) has char \( p \) \( \Rightarrow \mathbb{F}_p \cong K \)

\( K \) has char 0 \( \Rightarrow \mathbb{Q} \cong K \).
Recall: If $R$ is any ring and $F$ is a field, then any map $F \rightarrow R$ of rings is injective (ker is a ideal $\neq F \Rightarrow \ker = \{0\}$).

Remark/Exercise: If $K$ has characteristic $p$, then for any $a \in K$

\[ \underbrace{a + a + \ldots + a}_{p \text{ times}} = 0. \]

\[ p(a) = \left(1 + 1 + \ldots + 1\right) a. \]

\[ = 0 \cdot a = 0. \]

If $K \subseteq L$ are both fields, we say $K$ is a subfield of $L$ or $L/K$ is an extension of fields.

Degree of a field extension $L/K$

\[ [L : K] = \dim_{K}(L) \text{ as a } K\text{-vector space}. \]

E.g. $[C : \mathbb{R}] = 2$ and $\mathbb{C}$ has basis $1, i$ as an $\mathbb{R}$-vector space.

This can be infinite. But for us, well just work with $\mathbb{N}$ if it's finite or $\mathbb{C}$ if it's not.
$L/K$ - how to build $L$ from $K$?

Idea: Just add in one element at a time.

Suppose $\alpha \in L$, I want to look at $K(\alpha)$ defined to be the smallest subfield of $L$ containing both $K$ and $\alpha$.

$K \subseteq K(\alpha) \subseteq L$.

(The intersection of all such)

The trick of everything: $K(\alpha)$ admits a single abstract description.

$K(\alpha)$ is an extension of $K$, not an subfield of $L$.

Observation: There is a unique map

$K + \mathbb{C} \to L$

sending $1 \to \alpha$

and $= \text{identity on } K$.

Kernel of this map: 0 or $(m_\alpha(\mathbb{C}))$

monic irreducible polynomial.

Make sure you don't work down to some ideal first.
Let $Ker$ be $0$

$k + J \rightarrow L$

$L$ extends uniquely.

$k(c) \rightarrow L \quad k(c) = \mathbb{K}(\alpha) \leq L.$

$+ \rightarrow \alpha$

If $Ker = (M_{\alpha}(c))$

$k + J \rightarrow \frac{M_{\alpha}(c)}{M_{\alpha}(c)} \rightarrow L

$\leftarrow \text{maximal ideal}$

$\text{Image is } K(\alpha). \quad \text{so quotient is a field}$

$k + J / (M_{\alpha}(c)) \rightarrow \mathbb{K}(\alpha) \leq L.$

**Example:**

$R \in \mathbb{C}$

$+ \in \mathbb{C}.$

$R + J \rightarrow R(i)$

$+ \rightarrow i$

$\text{Kernel is } (t^2 + 1)$

$R(t^2 + 1) \rightarrow R(i) = \mathbb{C}.$

$R + J / (t^2 + 1) \rightarrow \mathbb{C}$

**Warning:** There is another construction like this.

$R + J / (t^2 + 1) \rightarrow \mathbb{C}$

Identify on $R$.

$t \rightarrow \text{root of } t^2 + 1.$

$t \rightarrow i$ (just saw)

$t + i \rightarrow \text{add and}$
$X$-algebras \xrightarrow{\text{Forget}} \text{Sets}$

\[ \text{Hom}(K_{E_{II}} \coprod_{I_{II}} I_{II}, A) = \text{Hom}(I_{II}, A) \]