Last time: characters, Fourier theory (on S^1)

2 additions:

1) If $\chi: G \to L^\times$ a character of G, L a field then $L^\times$ is abelian.

Factors as $G \to G^{ab} \to L^\times$

$$G^{ab} = G/[G,G]$$

2) Fourier theory on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$

f on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. (write integration over $0, 2\pi$)

$$\hat{f}(k) = \frac{\int_0^{2\pi} f(t) e^{-ikt}\, dt}{2\pi}$$

$k \in \mathbb{Z}$

the coefficient of $t \mapsto e^{ikt}$ in the expansion.

$$f(t) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikt}.$$

$$\hat{f}(k) = \langle f, e^{ikt} \rangle$$

Fourier theory on $\mathbb{Z}/n\mathbb{Z}$:

Recall: $X(\mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^*)$

$$\begin{array}{ccc} X(\mathbb{Z}/n\mathbb{Z}) & & t \mapsto e^{kt\,2\pi i/n} = \chi_k \\ \wr\!\| & & \updownarrow \\ \mathbb{Z}/n\mathbb{Z} & \ni & k \end{array}$$

$$\chi_k \in \underline{F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})} = \text{all functions from } \mathbb{Z}/n\mathbb{Z} \text{ to } \mathbb{C}.$$

$\mathbb{C}$ - vector space. n-dim'l.

$k \in \mathbb{Z}/n\mathbb{Z}$ $\quad \delta_k(t) = \begin{cases} 1 & \text{if } k = t \\ 0 & \text{if } k \neq t \end{cases}$.

If $f \in F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})$.

$$f = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} f(k)\,\delta_k.$$

a basis

Hermitian Inner Product on $F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}) = L^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})$

$$\langle f, g \rangle = \sum_{t \in \mathbb{Z}/n\mathbb{Z}} f(t)\,\overline{g}(t)$$

δ_k are an orthonormal basis.

$$(f, g) = \langle f, g \rangle / n$$

<u>Theorem</u> χ_k are an orthonormal basis for $(F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}), (\,,\,))$.

<u>Proof</u> There are n χ_k $= \dim F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})$. so suffices to show orthonormal.

Need to check $(\chi_{k_1}, \chi_{k_2}) = \begin{cases} 0 & \text{if } k_1 \neq k_2 \\ 1 & \text{if } k_1 = k_2. \end{cases}$

<u>Observation:</u> $\overline{\chi_k} = \dfrac{1}{\chi_k} = \chi_{-k}$.

Pf: $\overline{\chi_k}(t) = \overline{\chi_k(t)}$

Pf. $\overline{\chi_K(t)} = \overline{\ldots}$

$$= \overline{\left(e^{Kt2\pi i/n}\right)}$$

$$= e^{-Kt2\pi i/n} = \chi_{-K}(t)$$

$$(\chi_{K_1}, \chi_{K_2}) = \sum_{t\in\mathbb{Z}/n\mathbb{Z}} \frac{\chi_{K_1}(t)\overline{\chi_{K_2}(t)}}{n}$$

$$= \sum_{t\in\mathbb{Z}/n\mathbb{Z}} \frac{\chi_{K_1}(t)\chi_{-K_2}(t)}{n}$$

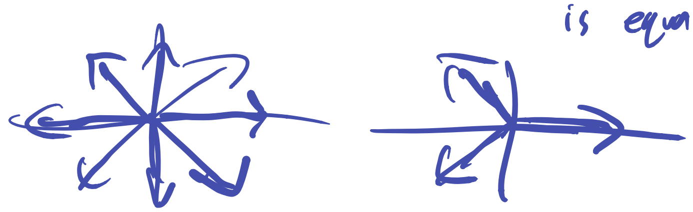
$$= \sum_{t\in\mathbb{Z}/n\mathbb{Z}} \frac{\chi_{(K_1-K_2)}(t)}{n}.$$

Suffices to see that

$$\sum_{t\in\mathbb{Z}/n\mathbb{Z}} \chi_K(t) = \begin{cases} 0 & \text{if } K \neq 0 \\ n & \text{if } K = 0. \end{cases}$$

$\chi_0(t) = 1$ for every t so ✓

Reduce to: For any $m > 1$ the sum of the mth roots of unity in $\mathbb{C}$ is equal to 0.



$$\prod_{K=0}^{m-1} \left(x - e^{2\pi i K/m}\right) = x^m - 1$$

coefficient of x^{m-1}

$= -\sum$ roots

$= -\sum_{k=0}^{m-1} e^{2\pi i k/m}$

$= 0$ if $m > 1$.

So: $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$.

$$f = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \hat{f}(k) \chi_k.$$

$$\hat{f}(k) = (f, \chi_k).$$

$$= \sum_{t \in \mathbb{Z}/n\mathbb{Z}} f(t) \overline{\chi_k(t)}$$

$$= \frac{\sum_{t \in \mathbb{Z}/n\mathbb{Z}} f(t) e^{-kt \frac{2\pi i}{n}}}{n}.$$

Remarks: · Generalizes easily to any finite abelian group.

· Generalizes to finite groups, but need higher dim'l rep. theory

---

$\mathbb{Z}/n\mathbb{Z} \circlearrowright F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})$ unitary action by right translation.

$$(j \cdot f)(t) = f(t+j).$$

Observation: For each character χ

$f \chi \quad F(\mathbb{Z}/ \quad t)$

$\mathbb{C}\cdot\chi \subseteq F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})$

is preserved by the group action

$$(j\cdot\chi)(t) = \chi(t+j) = \chi(t)\chi(j)$$
$$= \chi(j)\chi(t)$$

i.e.,

$$j\cdot\chi = \chi(j)\,(\chi)$$

The basis χ_k $k \in \mathbb{Z}/n\mathbb{Z}$.

$$F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}) = \bigoplus_{k\in\mathbb{Z}/n\mathbb{Z}} \mathbb{C}\chi_k.$$

This decomposition is preserved by the action of $\mathbb{Z}/n\mathbb{Z}$

Exercise: If χ is a character of G

$\rho: G \to GL(V)$ V $\mathbb{C}$-vector space.

if $v \in V$

$$v_\chi := \sum_{g\in G} \frac{1}{\chi(g)}\rho(g)(v)$$

satisfies $\rho(g)(v_\chi) = \chi(g)\,v_\chi$.

Theorem (Independence of characters):

If G is a group and L is a field, and $\chi_1, \ldots, \chi_k$ are distinct characters of G with values in L then they are linearly independent.

(in $F(G, L)$ ← L vector space of functions from G to L)

$(g \cdot f)(t) = f(tg)$

$g \cdot \chi = \chi(g)\chi$ ← $(g \cdot \chi)(t) = \chi(tg) = \chi(t)\chi(g)$

i.e: if $a_1\chi_1(g) + a_2\chi_2(g) + \cdots + a_k\chi_k(g) = 0$ for all $g \in G$ fixed $a_1, \ldots, a_k \in L$

then $a_1 = a_2 = \ldots = a_k = 0$.

Proof: Suppose not, take a non-trivial linear dependence involving as few as possible.

reordering, renumbering can assume

$$a_1\chi_1 + \ldots + a_k\chi_k = 0$$

$a_i \neq 0 \quad 1 \leq i \leq k.$

and no shorter relation.

For any g in G get.

$$a_1 g \cdot \chi_1 + \ldots + a_k g \cdot \chi_k = 0.$$

$$(a_1 \chi_1(g))\chi_1 + \cdots + (a_k \chi_k(g))\chi_k = 0.$$

Exercise: Get a contradiction by using this to produce a shorter non-trivial linear dependence. □