Announcements:

- HW for week 6 due March 16.
- Results for project due March 4th @ 11:59 pm, ON GRADESCOPE
- Exam on March 4th. - In class or Zoom. Contact if not possible.
- On March 2nd: Review A & A.

This week: Bridge between the first two record values of the class.

Characters:

Let $G$ be a group and $L$ a field.

$\text{(eg } G = \mathbb{Z}/n\mathbb{Z}, \ L = \mathbb{C}).$

A character of $G$ (with values in $L$)

is a homomorphism

$G \rightarrow L^* = GL_1(L) = L\setminus \{0\}.$

Exercise: What are the character of $\mathbb{Z}/n\mathbb{Z}$ with values in $\mathbb{C}$?

$\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}^*$

Need: $1 \in \mathbb{Z}/n\mathbb{Z}$ maps to an nth root of unity.

Why? $n \cdot 1 = 0.$
\[ \chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^x \]

\[ \chi(0) = 1 \]

\[ \chi(n-1) = 1 \]

\[ \chi(1)^n = 1 \]

So \( \chi(1) \) is an \( n \)-th root of unity.

Conversely: For any \( n \)-th root of unity \( z \)

\[ \exists! \quad \chi_z : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^x \]

\[ \chi_z(1) = z \]

\( n \)-th roots of unity in \( \mathbb{C}^x \):

Suppose \( z^n = 1 \), \( z \in \mathbb{C} \)

Then \( |z^n| = |1| = 1 \) so \( |z| = 1 \).

Thus \( z \in S^1 \) = complex numbers with absolute value 1.

Since \( z \in S^1 \)

\[ \exists! \quad \theta \in [0, 2\pi) \]

such that

\[ z = \cos \theta + i \sin \theta \]

\[ = e^{i\theta} \quad \text{(Euler's identity)} \]

\[ e^{z_1 + z_2} = e^{z_1} e^{z_2} \quad \text{for any } z_1, z_2 \text{ in } \mathbb{C} \]

\[ (e^{i\theta})^n = e^{in\theta} = 1 \]

\[ \iff n\theta \text{ is a multiple of } 2\pi \]

So \( z^n = 1 \iff z = e^{2\pi ik/n} \)

for \( 0 \leq k \leq n-1 \)
\[ X(\mathbb{Z}/n\mathbb{Z}) = \mathbb{C} \text{-valued characters of } \mathbb{Z}/n\mathbb{Z} \]

This is a group: \( (XG) \text{ for } G \)

\[ (\chi_1 \chi_2)(x) = \chi_1(x) \chi_2(x) \]

\[ X(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \]

\[ \chi_k(x) = e^{\frac{2\pi i x k}{n}} \quad \longleftrightarrow \quad k \]

**Example**

Continuous character of \( S^1 = \{ \exp(i \theta), |\theta| \leq 1 \} \).

Can show: any \( \chi: S^1 \rightarrow \mathbb{C}^\times \) factors through \( S^1 \) and is of the form \( 2^\mathbb{Z} \).

\[ X(S^1) \cong \mathbb{Z} \]

\( 2^\mathbb{Z} \leftrightarrow n \)

**Example**

Continuous character of \( \mathbb{R} \)

All of the form

\[ t \rightarrow e^{\lambda t} \quad \text{for } \lambda \in \mathbb{C} \]

\[ X(\mathbb{R}) = (\mathbb{C}, +) \]

\[ \sum e^{\lambda_1 t} e^{\lambda_2 t} = e^{(t, t)(\lambda_1, \lambda_2)} \]

**Fourier Theory:**
Periodic function on $\mathbb{R}$

Decomposes into its fundamental waves:
2 sin $\omega$ cos $\omega$
$\lambda \sin(\lambda x) \cos(\lambda x)$

Express a periodic function as
$\sum_{k=0}^{\infty} a_k \sin(\omega_k x) + b_k \cos(\omega_k x)$.

Signal analysis (Harmonic analysis).

What does this have to do with characters?

1. Periodic function $\rightarrow$ Function on $S^1$

Periodic function on $\mathbb{R}/2\pi \mathbb{Z} = S^1$
t $\rightarrow$ $e^{i\lambda t}$

Real valued functions $\rightarrow$ Complex valued functions
$\cos(\lambda \theta) + i \sin(\lambda \theta) = e^{i\lambda \theta}$

$\frac{e^{-i\lambda t} + e^{i\lambda t}}{2} = \cos(\lambda t)$

$\frac{e^{i\lambda t} - i e^{-i\lambda t}}{2} = \sin(\lambda t)$

$t \rightarrow e^{i\lambda t}$ as a function on $S^1$. 
Fourier Theory for $S^1$:
$L^2(S^1, \mathbb{C})$ is a vector space with inner product
\[
\langle f, g \rangle = \int_{S^1} f(z)\overline{g(z)} \, dz
\]
Any $f \in L^2(S^1, \mathbb{C})$ can be written as
\[
f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \quad \text{with} \quad |a_n| < \infty.
\]
A calculus exercise to prove.
\[
\sum_{n \in \mathbb{Z}} |a_n|^2 < \infty.
\]
A different perspective

$L^2(S^1, \mathbb{C})$ or $C^0(S^1, \mathbb{C})$

Spaces of functions on $S^1$.

$F$ = some space of functions on $S^1$.

$S^1$ acts on $F$ by right translation

$(z \cdot f)(x) = f(xz)$.

(Could replace $S^1$ with any group.)

$F$ is a $C^*$-module space.

$S^1 \cdot F$ is by linear operators

This a "representation" of $S^1$.

Can be viewed as an infinite dimensional vector space.

With inner product preserved by $S^1$.

This representation decomposes as a direct sum of characters.

$L^2(S^1) = \bigoplus_{n \in \mathbb{Z}} C^0(S^1; \mathbb{C})$.
The text on the page is not legible due to the handwriting style. It appears to be a mathematical expression or equation, possibly involving variables or functions, but the specific content cannot be accurately transcribed.