Recall

Last time — Finished proof of Sylow theorem.
  Simplicity of $A_n$ for $n \geq 5$.

Example: Groups of order 15.
  Talk about generalizing to groups of order $pq$.

Today: semidirect products.
  A tool for building new groups out of old groups.

Example: $D_8$, symmetries of the square.

$\mathbb{Z}/2\mathbb{Z} \leq D_8$
  $\uparrow$ Rotation

$\langle s \rangle$: Rotation by $\frac{k\pi}{2}$.
  $\langle s \rangle$: Reflection along $l$.

$\langle s \rangle = \mathbb{Z}/2\mathbb{Z} \leq D_8$.

$\langle s \rangle < \langle s \rangle = D_8$

We know $|HK| = \frac{|H||K|}{|H \cap K|}$.

($H$ is normal)

$\langle HK : H \rangle = \langle K : H \cap K \rangle$.

$HI/H \cong K/HK$. 
We can get every element of the group by multiplying elements of \( \mathbb{Z}/17 \times \mathbb{Z}/17 \).

i.e. there is a bijection of sets.

\[
\mathbb{Z}/17 \times \mathbb{Z}/17 \rightarrow D_8
\]

\[
K \times J \rightarrow \rho^K \rho^J
\]

Not a group isomorphism.

Semi-direct product: change the group law on the set \( \mathbb{Z}/17 \times \mathbb{Z}/17 \).

So that this becomes a group isomorphism.

\[
(K, J) \times (K', J') \mapsto \rho^K \rho^J \rho^{K'} \rho^{-J'}
\]
The example says

\[ D_8 \cong \mathbb{Z}/4\mathbb{Z} \times_{\phi} \mathbb{Z}/2\mathbb{Z} \]

\[ \phi: \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(\mathbb{Z}/4\mathbb{Z}) \]

\[ 1 \mapsto \text{multiplication by } -1. \]

**Definition/Theorem:**

If \( H \) and \( K \) are groups and \( \phi: K \to \text{Aut}_{\text{group}}(H) \), then \( H \times_{\phi} K \) is the set

\[ (h_1, k_1) \times_{\phi} (h_2, k_2) = (h_1, \phi(k_1)(h_2), K, K_2). \]

This defines a group, and

\[ H \to H \times_{\phi} K \]

\[ h \mapsto (h, 0) \]

identifies \( H \) with a normal subgroup of \( H \times_{\phi} K \).

\[ K \to H \times_{\phi} K \]

\[ k \mapsto (0, K) \]

identifies \( K \) with a subgroup.
s.t. \( KHK^{-1} = \phi(K)(h) \)

\[
(0, K) \phi(h, 0) \phi(0, K^{-1}) = (\phi(K)(h), 0).
\]

Example: \( D_{2n} = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) – a reflection. 
Symmetry of regular \( n \)-gon.

Exercise: what is \( \phi \)?

Example: For any group \( H \),
\[
H \times \text{Aut}(H).
\]

Recognition principle: If \( G \) is a group,
\( H \leq G \), \( K \leq N_G(H) \)
then \( HK \) is a subgroup of \( G \)
and if \( HNKH = \{e\} \), then the map
\( (h, K) \mapsto HK \)
is a group isomorphism
\[
H \times K \overset{\phi}{\to} HK
\]
(\( \phi : K \to N_G(H) \cong \text{Aut}(H) \)).

(Usually use Hall when \( HK = \text{one of } H \) or \( G \).

Groups of order \( pq \) For \( p \neq q \) prime:

Suppose \( G \) has order \( pq \).
\( p, q \) are primes.
\( p = 1 \mod {p} \) and \( p = 1 \mod {q} \).
\( p = 1, p+1, 2p+1, \ldots \) \( \mathbb{Z} / p \mathbb{Z} \)
so \( n \neq 1 \), so let \( H \triangleleft G \).

By Cauchy's Theorem, a subgroup of order \( q \), \( K \),

\[ HK = \mathbb{Z}/q\mathbb{Z} \]

\[ |HK| = p^2 \]

so \( HK = G \).

Recognition principle: \( G \cong H \times K \)

\[ \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \]

\[ \phi: \mathbb{Z}/p\mathbb{Z} \rightarrow (\mathbb{Z}/p\mathbb{Z})^* = \mathbb{Z}/(p-1)\mathbb{Z} \]

\[ \text{Aut}_{\text{group}}(\mathbb{Z}/p\mathbb{Z}) \]

If \( q \nmid p-1 \) then \( \phi \) is only non-trivial

\[ \Rightarrow G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \]

If \( q \mid p-1 \) then there is a unique cyclic subgroup of order \( q \) in \( \mathbb{Z}/(p-1)\mathbb{Z} \)

There will be \( q-1 \) non-trivial \( \phi \).

\[ \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{Z}/(p-1)\mathbb{Z} \]

Exercise: Check that \( \phi \) also gives isomorphic groups.

Important: Often \( \phi_1 \neq \phi_2 \)

but \( H \cdot K \cong H \cdot K \)

Conclusion: \( q \nmid p-1 \)
Then \( \mathbb{Z}/p\mathbb{Z} \) is the only group of order \( p \).

Otherwise there is \( \mathbb{Z}/p\mathbb{Z} \) and one nonabelian group of order \( p^2 \) (up to isomorphism).